

Sheaves and Local Subgroupoids

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University of Wales, Bangor, Mathematics Preprint 00.16

Abstract:

This is an introduction to the notion of local subgroupoid introduced by the author and R. Brown. It can also serve as an introduction to an application of sheaf theory, and so could be useful to beginners in that theory.

The main results are the construction of the holonomy groupoid and the notion of s -sheaf for certain local subgroupoids s .

2000 Mathematics Subject Classification: 18F20, 18F05, 58H05, 22A22.

Key Words: sheaf, section, groupoid, local equivalence realtions, local subgroupoids, holonomy groupoid, s -sheaf.

Contents

1	Sheaves, Atlases and Groupoids	3
1.1	Presheaves and Sheaves of Sets	3
1.2	Direct and Inverse Image	10
1.3	Chart and Atlas	11
1.4	Local equivalence relations	13
1.5	The Category of Groupoids	17
1.5.1	Examples	19
2	Local and Global Subgroupoids	23
2.0.2	Local subgroupoids	23
2.0.3	Examples	24
2.1	Coherent Local Subgroupoids	29
2.1.1	Topological foliations	34
3	Holonomy groupoid	36
3.1	Local subgroupoids and Locally Topological Groupoids	38
3.2	Holonomy groupoid	41
4	s-sheaves	44
4.1	Internal Category	44
4.2	G-sheaves	47
4.3	r-sheaf	49
4.4	s-sheaf	50

Introduction

Local to global problems play a very important role in mathematics. The most important concept in this context is a sheaf on a topological space. A sheaf is a way of describing a class of functions, sets, groups, etc. For instance, a class of continuous functions on a topological space X is very important in sheaf theory. The description tells the way in which a function f defined on an open subset U of X can be restricted to functions $f|_V$ on open subsets $V \subseteq U$ and then can be recovered by piecing together the restrictions to the open subsets. This applies not just to functions, but also to other mathematical structures defined ‘locally’ on a space X , for example, see [35, 36, 37].

Chapter 1 gives an exposition of some needed preliminaries and facts of the basic concepts on sheaves, presheaves. An important notion in sheaf theory is that of a global section of a sheaf. We show how such sections can be described explicitly in term of *atlases*. This notion is important for all our later work. Also Chapter 1 reviews the concept of local equivalence relations, which was introduced by Grothendieck and Verdier [26] in a series of exercises presented as open problems concerning the construction of a certain kind of topos was investigated further by Rosenthal [51, 52] and more recently by Kock and Moerdijk [38, 39]. A local equivalence relation is a global section of the sheaf \mathcal{E} which is defined by the presheaf

$$E = \{E(U), E_{UV}, X\},$$

where $E(U)$ is the set of all equivalence relations on the open subset U of X and E_{UV} is the restriction map from $E(U)$ to $E(V)$, for $V \subseteq U$. Simple examples show that this presheaf is in general not a sheaf. It is also becoming appaerant that groupoids are another important tool in local-to-global problems. Therefore in Chapter 1 we also describe basic concepts of groupoids.

Chapter 2 introduce the recent idea of a local subgroupoid of a groupoid G on a topological space X as a global section of the sheaf \mathcal{L} associated to the presheaf

$$L_G = \{L(U), L_{UV}, X\}$$

where $L(U)$ is the set of all wide subgroupoids of $G|_U$ and L_{UV} is the restriction map from $L(U)$ to $L(V)$ for $V \subseteq U$. The idea of transitive connectness is important in the theory and examples for local subgroupoids. We also introduce notions of coherence which allows for an adjoint functional relationship between local and global subgroupoids and obtain a topological foliation from a local subgroupoid.

Chapter 3 defines the holonomy groupoid of certain local subgroupoid by using the idea of locally topological groupoid. We define a strictly regular local subgroupoid s and prove that if s is a strictly regular local subgroupoid of the topological groupoid G on X and

$$glob(s) = H, \quad W = \bigcup_{x \in X} H_x,$$

then (H, W) admits the structure of a locally topological groupoid. So we obtain a holonomy groupoid H^s of the strictly regular local subgroupoid s .

Chapter 4 introduce the concept of s -sheaves for strictly regular local subgroupoids s . Corresponding concept for local equivalence relation r was extensively investigated by Rosental [51, 52] and Kock and Moerdjik [38, 39] where they show that the r -sheaves form an étendue. This still leaves as an open problem that of describing the kind of topos formed by the category of s -sheaves.

Acknowledgements: I would like to thanks to Prof. Ronald Brown, for his suggestings, help and encouragement in all stages of the preparation of this work and of my PhD thesis [32].

Chapter 1

Sheaves, Atlases and Groupoids

1.1 Presheaves and Sheaves of Sets

Let X be a topological space and let $\mathcal{O}(X)$ be the set of open subsets of X . The set $\mathcal{O}(X)$ is partially ordered by inclusion, so we can regard it as a small category in the usual sense, i.e., the objects of $\mathcal{O}(X)$ are the open sets in X , and its morphisms are the inclusion maps. Also we can form a new category $\mathcal{O}(X)^{op}$, called the opposite or dual category of $\mathcal{O}(X)$, by taking the same objects but reversing the direction of all the morphism and the order of all compositions. In other word , an arrow $V \rightarrow U$ in $\mathcal{O}(X)^{op}$ is the same thing as an arrow $U \rightarrow V$ in $\mathcal{O}(X)$.

Definition 1.1.1 Let X be a topological space. A *presheaf* F of sets on X is given by the following pieces of information;

- (i) for each open set U of X , a set $F(U)$,
- (ii) for each inclusion of open sets $V \subseteq U$ of X , a restriction map $F_{UV} : F(U) \rightarrow F(V)$ such that
 1. $F_{UU} = id_U$ 2. $F_{VW} \circ F_{UV} = F_{UW}$ whenever $W \subseteq V \subseteq U$.

Thus, using functorial terminology we have the following definition. Let X be a topological space. A *presheaf* F on X is a functor from the category $\mathcal{O}(X)^{op}$ of open subset of X and inclusions to the category *Sets* of sets and functions:

$$F : \mathcal{O}(X)^{op} \rightarrow Sets$$

Then the system $F = \{F(U), F_{UV}, X\}$ is said to be a *presheaf* of sets on X .

In general, we define a presheaf with values in an arbitrary category. For example, if the presheaf satisfies the following properties, it is said to be a presheaf of \mathbb{R} -algebras:

- (i) every $F(U)$ is an \mathbb{R} -algebra,
- (ii) for $V \subseteq U$, $F_{UV} : F(U) \rightarrow F(V)$ is an \mathbb{R} - *algebra* homomorphism.

That is, F is a functor from $\mathcal{O}(X)^{op}$ to the category $\mathbb{R} - Alg$ of the \mathbb{R} -algebra and \mathbb{R} -algebra homomorphisms: $F : \mathcal{O}(X)^{op} \rightarrow \mathbb{R} - Alg$ [34].

Examples of presheaves are abundant in mathematics. For instance, if A is an abelian group, then there is the *constant functor* F with $F(U) = A$ for all open U and $F_{UV} = id_A$ for all $V \subseteq U$. This functor defines the *constant presheaf*. We also have the presheaf F assigning to U the group (under pointwise addition) $F(U)$ of all function from U to A , where F_{UV} is the canonical restriction, i.e., a functor $F : \mathcal{O}(X)^{op} \rightarrow Grp$. If $A = \mathbb{R}$ we also have the presheaf $R_{\mathbb{R}}$ with $F(U)$ being the group of all continuous real-valued functions on U . Similarly, we have the presheaves of differentiable functions on (open subsets of) a differentiable manifold X ; of differential p - *forms* on X ; of vector field on X ; and so on. In algebraic topology, we have good presheaf examples: the presheaf of singular p -cochains of open subsets $U \subseteq X$; the presheaf assigning to U its p th singular cohomology group; the presheaf assigning to U the p th singular chain group of $X \bmod X - U$; and so on. For more examples, see [53, 5, 54, 36, 37, 44].

Let us consider a presheaf F on a topological space as follows:

$$F : \mathcal{O}(X)^{op} \rightarrow Sets.$$

Let x be point in the topological space X and let U, V be two open neighbourhoods of x and let $s \in F(U), t \in F(V)$. So let us consider the set

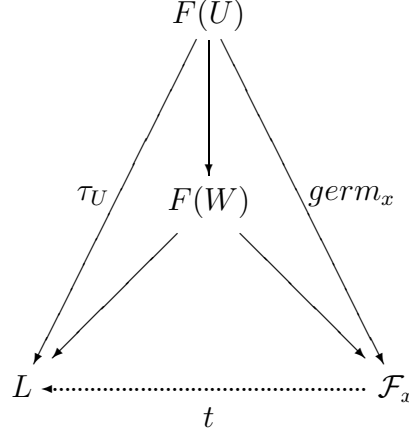
$$M = \{(U, s) : U \text{ is open in } X, s \in F(U)\}.$$

We can define an equivalence relation on M as follows; we say that s and t have the *same germ at x* when there is some open set $W \subseteq U \cap V$ with $x \in W$ and $s|_W = t|_W \in F(W)$. The relation *has the same germ at x* is an equivalence relation on M , and the equivalence class of any one such s is called the *germ* of s at x , in symbols $germ_x s$. Let

$$\mathcal{F}_x = \{(U, s)_x = germ_x s \mid s \in F(U), x \in U \text{ open in } X\}$$

be the set of all germs at x . Then, letting $F^{(x)}$ be the restriction of the functor $F : \mathcal{O}(X)^{op} \rightarrow Sets$ to open neighbourhoods of x , the function $germ_x : F(U) \rightarrow \mathcal{F}_x$ forms a cone on $F^{(x)}$ as on the right of the figure below (because $germ_x s = germ_x(s|_W)$ whenever $x \in W \subseteq U$ and $s \in F(U)$) [44].

Also, if $\{\tau_U : F(U) \rightarrow L\}_{x \in U}$ on the last below is any other cone over $F^{(x)}$, the definition of *same germ* implies that there is a unique function $t : \mathcal{F}_x \rightarrow L$, with $t \circ \text{germ}_x = \tau$.



This just states in detail that the set \mathcal{F}_x is the colimit and germ_x is the colimiting cone of the functor F restricted to open neighbourhoods of x :

$$\mathcal{F}_x = \lim_{\substack{\longrightarrow \\ x \in U}} F(U).$$

This statement summarizes the definition of *germ*. The set \mathcal{F}_x of all germs at x is usually called the *stalk* of P at x . Now combine the various sets \mathcal{F}_x of germs in the disjoint union \mathcal{F} (over $x \in D$).

$$\mathcal{F} = \bigcup_{x \in X} \mathcal{F}_x = \bigcup_{x \in X} \{(U, s)_x = \text{germ}_x s \mid x \in U \subseteq X, \text{open}, s \in F(U)\}$$

and define a canonical projection $p : \mathcal{F} \rightarrow X$ as the map sending each $\text{germ}_x s = (U, s)_x$ to the point x , i.e., $p(\mathcal{F}_x) = x$.

The set \mathcal{F} will be provided with a topology such that p becomes a local homeomorphism. Let $U \subseteq X$ open and $s \in F(U)$. Then each $s \in F(U)$ determines a function \dot{s} by

$$\dot{s} : U \rightarrow \mathcal{F}, \quad \dot{s}(x) = (U, s)_x, \quad x \in U.$$

We also define

$$\dot{s}(U) = \bigcup_{x \in U} (U, s)_x.$$

Topologise this set \mathcal{F} by taking as a base of open sets all the images $\dot{s}(U) \subseteq \mathcal{F}$, i.e., the family

$$\mathcal{T} = \{\dot{s}(U) \mid U \subseteq X \text{ open}, s \in F(U)\}$$

defines a topological base on \mathcal{F} .

Let $\dot{s}_1(U_1), \dot{s}_2(U_2) \in \mathcal{T}$. If $\dot{s}_1(U_1) \cap \dot{s}_2(U_2) = \emptyset$, then $\emptyset \in \mathcal{T}$, since $\dot{s}(\emptyset) = \cup_{x \in \emptyset} (\emptyset, s)_x = \emptyset$. Suppose that $\dot{s}_1(U_1) \cap \dot{s}_2(U_2) \neq \emptyset$. Then there is an element $\sigma \in \dot{s}_1(U_1) \cap \dot{s}_2(U_2)$ such that $p(\sigma) =$

$x \in U_1 \cap U_2$. This gives an open neighbourhood of $x \in U \subseteq U_1 \cap U_2$ such that $\sigma = \dot{s}_2(x) = \dot{s}_1(x)$. For every $x \in U$, since $\dot{s}(x) = \dot{s}_1(x)$, $\dot{s}(U) = \dot{s}_1(U) \subseteq \dot{s}_1(U_1) \cap \dot{s}_2(U_2)$. So $\dot{s}_i(U)$ lies $\dot{s}_1(U_1) \cap \dot{s}_2(U_2)$, for $i = 1, 2$ and σ is an interior point of $\dot{s}_1(U_1) \cap \dot{s}_2(U_2)$ [11].

Hence \mathcal{F} is a topological space with the above topology. This topology is called the *sheaf topology* on \mathcal{F} .

Now we have to show that p is a local homeomorphism with this topology, i.e., for each $\sigma = (U, s)_x \in \mathcal{F}$, $x \in X$, there are open sets U, W with $\sigma \in W \subseteq \mathcal{F}$ and $p(\sigma) = x \in U \subseteq X$ such that $p|_W; W \rightarrow U$ is a homeomorphism, whereas for $\sigma = (U, s)_x \in \mathcal{F}$, $p(\sigma) = p((U, s))_x = x$. Let $\dot{s} : U \rightarrow \mathcal{F}$, $\dot{s}(x) = (U, s)_x = \sigma \in \mathcal{F}_x$, for $x \in U$.

Let $W = \dot{s}(U)$ and $p|_U = p'$.

Firstly we will show that p' is bijective. In fact, for $\sigma_1, \sigma_2 \in \dot{s}(U) = W$, there are two elements $x_1, x_2 \in U$ such that $\sigma_1 = \dot{s}(x_1)$ and $\sigma_2 = \dot{s}(x_2)$. If $p'(\sigma_1) = p'(\sigma_2)$, then $p'(\sigma_1) = p'(\dot{s}(x_1)) = p'(\sigma_2) = p'(\dot{s}(x_2)) = x_1 = x_2$. This implies $\dot{s}(x_1) = \dot{s}(x_2)$, i.e., $\sigma_1 = \sigma_2$.

The map p' is continuous. Choose any point $\sigma \in W = \dot{s}(U)$ such that $p'(\sigma) = x \in U$. Then there is an open neighbourhood $x \in V \subseteq U$ such that $\dot{s}(V) \subseteq W = \dot{s}(U)$ is an open neighbourhood of σ and $p'(\dot{s}(V)) = V \subseteq U$. So p' is continuous.

Now we shall show that $p^{-1} = (p|_W)^{-1} = \dot{s} : U \rightarrow W = \dot{s}(U)$ is continuous. If an arbitrary element $x \in U$, $\dot{s}(x) = \sigma \in W$, $W' \subseteq W$ is an open neighbourhood of σ , then $(p|_W)(W') \subseteq U$ is an open neighbourhood of x in U and $\dot{s}(p|_W) = W'$. Hence \dot{s} continuous.

These facts lead us to the basic definition of a sheaf on a topological space X .

Definition 1.1.2 A *sheaf* on a topological space X is a pair (\mathcal{F}, p) , where

- (i) \mathcal{F} is a topological space (not Hausdorff in general, see, [11]).
- (ii) $p : \mathcal{F} \rightarrow X$ is a local homeomorphism.

Then we state following theorem.

Theorem 1.1.3 Every presheaf F on a topological space X defines a sheaf \mathcal{F} over X in the above manner.

Definition 1.1.4 Let \mathcal{F} be a sheaf on X . Let x be an arbitrary point in X and let U be an open neighbourhood of x . A *section* over U is a continuous map $\dot{s} : U \rightarrow \mathcal{F}$ such that $p \circ \dot{s} = id_U$. We denote the set of all sections of \mathcal{F} over U by $\Gamma(U, \mathcal{F})$.

The set of sections $\Gamma(U, \mathcal{F})$ gives a presheaf as follows. If $V \subseteq U$,

$$\Gamma_{UV} : \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(V, \mathcal{F}), \quad \dot{s} \mapsto \dot{s}|_V$$

is the restriction, so we get a functor

$$\Gamma : O(X)^{op} \longrightarrow Sets.$$

Hence

$$\Gamma = \{\Gamma(U, \mathcal{F}), \Gamma_{UV}, X\}$$

is a presheaf on X . This presheaf is called the *canonical presheaf*. The set of global sections of \mathcal{F} is given by $\Gamma(X, \mathcal{F})$. The presheaf Γ defines a sheaf $\Gamma\mathcal{F}$ over X by Theorem 1.1.3. Moreover every element $s \in \mathcal{F}U$ is associated with a section $\dot{s} \in \Gamma(U, \mathcal{F})$. If $x \in X$ and $\sigma \in \mathcal{F}_x$, then there are an open neighbourhood $x \in U \subseteq X$ and an $\dot{s} \in \Gamma(U, \mathcal{F})$ such that

$$\sigma = (U, H)_x = \dot{s}(x) = s_x.$$

We shall now list some elementary properties of the sheaf \mathcal{F} and the set of sections $\Gamma(U, \mathcal{F})$, for an open set $U \subseteq X$:

(i) p is an open map.

(ii) Let $\dot{s} : U \rightarrow \mathcal{F}$ be a map with $p \circ \dot{s} = id_U$, for an open set $U \subseteq X$. Then $\dot{s} \in \Gamma(U, \mathcal{F})$ if and only if \dot{s} is open.

(iii) Let U be open in X and $\dot{s} \in \Gamma(U, \mathcal{F})$. Then $p : \dot{s}(U) \rightarrow U$ is a homeomorphism and $\dot{s} = (p|_{\dot{s}(U)})^{-1}$.

(iv) Let σ be an arbitrary point in \mathcal{F} . Then there exists an open set $V \subseteq X$ and a section $\dot{s} \in \Gamma(V, \mathcal{F})$ with $\sigma \in \dot{s}(V)$.

(v) For any two sections $\dot{s}_1 \in \Gamma(U_1, \mathcal{F})$ and $\dot{s}_2 \in \Gamma(U_2, \mathcal{F})$, U_1 and U_2 opens, the set U of points $x \in U \subseteq U_1 \cap U_2$ such that $\dot{s}_1(x) = \dot{s}_2(x)$ is open.

Note that if \mathcal{F} were Hausdorff then the set U of (v) would also be closed in $U_1 \cap U_2$.

1). Let \mathcal{F} be a sheaf on a topological space X and let Γ the presheaf of sections of \mathcal{F} . The presheaf Γ defines a sheaf denoted by $\Gamma\mathcal{F}$. Clearly there is a natural map

$$\mathcal{F} \rightarrow \Gamma\mathcal{F}, \quad germ_x s \mapsto germ_x \dot{s}$$

which is a homeomorphism and preserves algebraic structure, if it has.

2). Let F be a presheaf with an algebraic structure (such as group, ring, etc.) and \mathcal{F} the sheaf that it generates. For such any open set $U \subseteq X$ there is a natural map

$$\mu_U : F(U) \rightarrow \Gamma(U, \mathcal{F}), \quad s \mapsto \dot{s}.$$

When is μ_U an isomorphism for all U [5]. Recalling that

$$\mathcal{F}_x = \lim_{\substack{\longrightarrow \\ x \in U}} F(U)$$

it follows that an element $s \in F(U)$ is in $Ker \mu_U$ if and only if s is *locally trivial* (that is, for every $x \in U$ there is a neighbourhood V of x such that $s|_V = 0$).

Thus μ_u is a monomorphism for all $U \subseteq X$ if and only if the following condition holds:

F₁ If $U = \cup U_i$, with U_i open in X , for $i \in I$, and $s, t \in F(U)$ are such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

Clearly, in F_1 we could assume that $t = 0$. However, the condition is phrased so that it applies to presheaf of sets.

Similarly, let $\dot{t} \in \Gamma(U, \mathcal{F})$. For each $x \in U$ there is a neighbourhood U of x and an element $t \in F(U)_i$ with $\mu_{U_i}(t)(x) = \dot{t}(x)$. Since $p : \mathcal{F} \rightarrow X$ is a local homeomorphism, $\mu(t)$ and \dot{t} coincide in some neighbourhood V of x . We may assume that $U = V$. Now $\mu(s_i|_{U_i \cap U_j}) = \mu(s_j|_{U_i \cap U_j})$ so that, if **F₁** holds, we obtain $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. If Γ were a presheaf sections (of any map) then this condition would imply that the s_i are restrictions to U_i of a section $s \in F(U)$. Conversely, if there is an element $s \in F(U)$ with $s|_{U_i} = s_i$ for all $i \in I$, then $\mu(s) = t$.

We have shown that, if **F₁** holds, then μ_U is surjective for all U and (hence an isomorphism) if and if the following condition is satisfied.

F₂ Let $\{U_i : i \in I\}$ be a collection of open sets in X and let $U = \cup U_i$, if $s_i \in F(U_i)$ are given such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j then there exists an element $s \in F(U)$ with $s|_{U_i} = s_i$ for all $i \in I$.

Thus, sheaves are in one to one correspondence with presheaves satisfying **F₁** and **F₂**. For this reason it is common practice not to distinguish between sheaves and presheaves satisfying **F₁** and **F₂**. Indeed, in certain generalisations of the theory, the Definition 1.0.2 is not available and the other notion is used. This will not be of concern to us.

Note that **F₁** and **F₂** are equivalent to the hypothesis that the following diagram (*) is an equalizer diagram

$$F(U) \xrightarrow{e} \prod_i F(U)_i \xrightleftharpoons[q]{p} \prod_{i,j} F(U_i \cap U_j). \quad (*)$$

So we can define a *sheaf* \mathcal{F} of sets on a topological space X as a functor $F : \mathcal{O}^p \rightarrow \text{Sets}$ such that each open covering $U = \cup U_i$, $i \in I$, of an open set U of X yields an equalizer diagram (*), where for $t \in F(U)$, $e(t) = \{t|_{U_i} : i \in I\}$ and for a family $t_i \in F(U)_i$, $w(t_i) = \{t_i|_{U_i \cap U_j}\}$, $q(t_i) = \{t_j|_{U_i \cap U_j}\}$ [44].

Definition 1.1.5 Let F_1 and F_2 be presheaves of sets on the topological space X . A presheaf morphism $h : F_1 \rightarrow F_2$ is a collection of morphism $h_U : F_1 U \rightarrow F_2 U$ commuting with restrictions: That is, h is a natural transformation: the diagram

$$\begin{array}{ccc} F_1(U) & \xrightarrow{h_U} & F_2(U) \\ F_{1UV} \downarrow & & \downarrow F_{2UV} \\ F_1(V) & \xrightarrow{h_V} & F_2(V) \end{array}$$

is commutative.

Definition 1.1.6 Let $(\mathcal{F}_1, p_1), (\mathcal{F}_2, p_2)$ be sheaves over X .

A function $\eta : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is called *stalk preserving* if $p_2 \circ \eta = p_1$ (therefore $\eta((\mathcal{F}_1)_x) \subseteq (\mathcal{F}_2)_x$ for all $x \in X$).

A *sheaf morphism* is a continuous stalk preserving function $\eta : \mathcal{F}_1 \rightarrow \mathcal{F}_2$.

A *sheaf isomorphism* is a stalk preserving homeomorphism $\eta : \mathcal{F}_1 \rightarrow \mathcal{F}_2$. The sheaves $\mathcal{F}_1, \mathcal{F}_2$ are called isomorphic if there exists a sheaf isomorphism between them.

The set $Sh(X)$ will denote the category of all sheaves \mathcal{F} of sets on the topological space X with these morphisms as arrows; so, by definition, $Sh(X)$ is a full subcategory of the functor category $Sets^{\mathcal{O}^{op}}$ [44].

Theorem 1.1.7 Let $(\mathcal{F}_1, p_1), (\mathcal{F}_2, p_2)$ be sheaves over X , and $\eta : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a stalk preserving map. Then the following statements are equivalent.

- i). $\eta : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is a sheaf morphism.
- ii). For every open $U \subseteq X$ and every section $\dot{s} \in \Gamma(U, \mathcal{F}_1)$, $\eta \circ \dot{s} \in \Gamma(U, \mathcal{F}_2)$.
- iii). For every element $\sigma \in \mathcal{F}_1$ there exist an open set $U \subseteq X$ and a section $\dot{s} \in \Gamma(U, \mathcal{F}_1)$ with $\sigma \in \dot{s}(U)$ and $\eta \circ \dot{s} \in \Gamma(U, \mathcal{F}_2)$.

Proof: If η is continuous, $U \subseteq X$ open and $\dot{s} \in \Gamma(U, \mathcal{F}_1)$ then $\eta \circ \dot{s}$ is also continuous. Moreover $p_2 \circ (\eta \circ p_1) = (p_2 \circ \eta) \circ \dot{s} = p_1 \circ \dot{s} = id_U$. Therefore $\eta \circ \dot{s}$ lies in $\Gamma(U, \mathcal{F}_2)$.

If $\sigma \in \mathcal{F}_1$, then there exists an open set $U \subseteq X$ and $\dot{s} \in \Gamma(U, \mathcal{F}_1)$ with $\sigma \in \dot{s}(U)$. If the condition of (ii) are also satisfied, then $\eta \circ \dot{s}$ lies in $\Gamma(U, \mathcal{F}_2)$

If for a given $\sigma \in \mathcal{F}_1$, open set $U \subseteq X$ and a $\dot{s} \in \Gamma(U, \mathcal{F}_1)$ with $\sigma \in \dot{s}(U)$ and $\eta \circ \dot{s} \in \Gamma(U, \mathcal{F}_2)$ are chosen according to condition (iii), then $\dot{s} : U \rightarrow \dot{s}(U)$ is a homeomorphism. Therefore $\eta|_{\dot{s}(U)} = (\eta \circ \dot{s}) \circ \dot{s}^{-1} : \dot{s}(U) \rightarrow \mathcal{F}_2$ is continuous and therefore η is continuous at σ . ■

Corollary 1.1.8 Every sheaf morphism is an open map.

Proof: Let $\eta : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a sheaf morphism. Since \mathcal{F}_1 is canonically isomorphic to the sheaf $\Gamma\mathcal{F}_1$ defined by its canonical presheaf $\Gamma_{\mathcal{F}_1}$. The sets $\dot{s}(U)$ with $\dot{s} \in \Gamma(U, \mathcal{F}_1)$ form a basis of the topology of \mathcal{F}_1 . If \dot{s} lies in $\Gamma(U, \mathcal{F}_1)$, then $\eta \circ \dot{s}$ lies in $\Gamma(U, \mathcal{F}_2)$, by theorem 1.1.7 (ii) and hence $\eta(\dot{s}(U)) = \eta \circ \dot{s}(U)$ is open in \mathcal{F}_2 . ■

Remark: For every open subset $U \subseteq X$, a sheaf morphism $\eta : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ defines a map

$$\eta_* : \Gamma(U, \mathcal{F}_1) \rightarrow \Gamma(U, \mathcal{F}_2)$$

by $\eta_*(s) = \eta \circ \dot{s}$.

In this stage we can define category of global sections of sheaves.

Category of Sections of Sheaves

Let $Sh(X)$ be the category of sheaves on a topological space X . We define a category of global sections of $Sh(X)$ which is denoted by Sec_X as follows;

In Sec_X , an arrow $\phi: s_1 \rightarrow s_2$ is a sheaf morphism $\phi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ with $s_1\phi = s_2$, i.e., the following diagram

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{\phi} & \mathcal{F}_2 \\ & \swarrow s_1 \quad \searrow s_2 & \\ & X & \end{array}$$

commutes. The set of objects, $Ob(Sec_x)$ is clearly global sections of $Sh(X)$.

1.2 Direct and Inverse Image

Definitions of direct and inverse image of sheaves can be found any sheaf theory book, see [53, 44, 36].

Let X, Y be topological spaces and let $f: X \rightarrow Y$ be a continuous map. Then each sheaf \mathcal{F} on X yields a sheaf $f_*\mathcal{F}$ on Y defined, for open set V in Y by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V)$; that is, $f_*\mathcal{F}$ is defined as the composition functor

$$\mathcal{O}(Y)^{op} \xrightarrow{f^{-1}} \mathcal{O}(X)^{op} \xrightarrow{\mathcal{F}} Sets$$

This sheaf $f_*\mathcal{F}$ is called the *direct image of \mathcal{F} under f* . The map f_* so defined is clearly a functor

$$f_*: Sh(X) \rightarrow Sh(Y).$$

Also $(fg)_* = f_*g_*$, so the definition $Sh(f) = f_*$ makes Sh a functor on the category of all topological spaces. In particular, if $f: X \rightarrow Y$ is a homeomorphism, f_* gives an isomorphism of categories between sheaves on X and on Y .

Now let \mathcal{F} be a sheaf on Y . The *inverse image* $f^*\mathcal{F}$ of \mathcal{F} is the sheaf on X defined by

$$f^*\mathcal{F} = \{(x, \sigma) \in X \times \mathcal{F} : f(x) = p(\sigma)\}$$

where $p: \mathcal{F} \rightarrow Y$ is the canonical projection of sheaf, i.e. p is a local homeomorphism. A projection on $f^*\mathcal{F}$ is given by

$$p^*: f^*\mathcal{F} \rightarrow X, \quad (x, \sigma) \mapsto x.$$

To check that $f^*\mathcal{F}$ is indeed a sheaf we note that if $U \subseteq Y$ is an open neighbourhood of $f(x)$ and $\dot{s}: U \rightarrow \mathcal{F}$ is a section of \mathcal{F} with $\dot{s}(f(x)) = \sigma$, then the neighbourhood $(f^{-1}(U)) \times \dot{s}(U) \cap f^*\mathcal{F}$ if

(x, σ) in $f^*\mathcal{F}$ is precisely

$$\{(x', \dot{s}f(x')) : x' \in f^{-1}(U)\}$$

and hence maps homeomorphically onto $f^{-1}(U)$.

Then, each continuous map $f : X \rightarrow Y$ gives a functor $f^* : Sh(Y) \rightarrow Sh(X)$. For a sheaf \mathcal{F} on Y , the value $f^*\mathcal{F} \in Sh(X)$ of this functor is called the *inverse image of \mathcal{F} under f* .

Theorem 1.2.1 *If $f : X \rightarrow Y$ is a continuous map, then the functor f^* , sending each sheaf \mathcal{F} on Y to its inverse image on X , is left adjoint to the direct image functor f_* ;*

$$Sh(X) \begin{matrix} \xrightarrow{f_*} \\ \xleftarrow{f^*} \end{matrix} Sh(Y)$$

Proof: See Mac Lane and Moerdijk [44]. ■

Definition 1.2.2 Let \mathcal{F} be a sheaf on X and $Y \subseteq X$. Then

$$\mathcal{F}|_Y = p^{-1}(Y)$$

is a sheaf on Y called the *restriction of \mathcal{F} on Y* .

Let F be a constant presheaf on X . A sheaf \mathcal{F} which is generated by F is called *constant sheaf*. In other word, the constant sheaf on X with stalk $F(U) = A$ is the sheaf $X \times A$ (giving A the discrete topology).

A sheaf \mathcal{F} on X is called *locally constant* if every point of X has a neighbourhood U such that $\mathcal{F}|_U$ is constant. If a presheaf F is a sheaf in the functorial terminology, then the locally constant sheaf can be defined as follows; A sheaf \mathcal{F} on a topological X is called *locally constant* if each point $x \in X$ has a basis of open neighbourhood N_x such that whenever $U, V \in N_x$ with $V \subseteq U$, the restriction

$$F_{UV} : F(U) \rightarrow F(V)$$

is a bijection. We also say that \mathcal{F} is *locally constant* if and only if $p : \mathcal{F} \rightarrow X$ is a covering [44].

We will give definition of *atlas* and *chart* due to [38], for any presheaf F or, more precisely, an *atlas* for a global section of the sheaf \mathcal{F} associated to a presheaf F .

1.3 Chart and Atlas

A *global section* of the sheaf \mathcal{F} associated to a presheaf F on a topological space X can be constructed in different ways. In sheaf theory, this is usually done by constructing the sheaf

space, consisting of germs of elements of F , at various points, but for the present purpose, a description in term of atlases is more appropriate [38].

Suppose we are given a presheaf F on a topological space X . An *atlas* in the presheaf F , or an atlas for a global section of \mathcal{F} , consists of a family,

$$\mathcal{U} = \{(U_i, s_i) : i \in I, s_i \in F(U_i)\}$$

such that

- (i) $X = \bigcup_{i \in I} U_i$, i.e., the family $\{U_i : i \in I\}$ is an open covering of X .
- (ii) (*Local compatibility Condition*). For all $i, j \in I$, there exist an open cover of $U_i \cap U_j$ by open sets W for which

$$s_i|_W = s_j|_W.$$

If \mathcal{U} is an atlas as above, then each (U_i, s_i) is called its *chart*.

Proposition 1.3.1 *Every global section s of the sheaf \mathcal{F} associated to presheaf F is given by some atlas. Conversely, every atlas in F determines a global section.*

Proof: Let

$$\mathcal{U} = \{(U_i, s_i) : i \in I, s_i \in F(U_i)\}$$

be an atlas for a presheaf $F : \mathcal{O}(X)^{op} \rightarrow Sets$.

We claim that the atlas \mathcal{U} defines a global section of the sheaf \mathcal{F} associated to the presheaf F above.

Clearly we can define an equivalence relation on the atlas \mathcal{U} as follows; let us fix an element $x \in X$. Let $(U_i, s_i), (U_j, s_j)$ be two elements of \mathcal{U} such that $x \in U_i \cap U_j$. We say that $(U_i, s_i), (U_j, s_j)$ are equivalent, written $(U_i, s_i) \sim_x (U_j, s_j)$ iff there is a neighbourhood W such that $x \in W \subseteq U_i \cap U_j$ and $s_i|_W = s_j|_W$. Let $(U_i, s_i)_x$ denote the equivalence classes of (U_i, s_i) . Then we obtain stalks and their sheaf as usual.

$$\mathcal{F}_x = \{(U_i, s_i)_x : x \in U_i, s_i \in F(U_i)\}, \quad \mathcal{F} = \bigcup_{x \in X} \mathcal{F}_x$$

So each $(U_i, s_i)_x$ determines a map \dot{s}_i by

$$\dot{s}_i : U_i \rightarrow \mathcal{F} \quad \dot{s}_i(x) = (U_i, s_i)_x \quad x \in U_i$$

such that \dot{s}_i is continuous in the sheaf topology of \mathcal{F} . Since \mathcal{U} is an atlas, then the formula

$$\dot{s}(x) = \dot{s}_i(x) \text{ for } x \in U_i.$$

defines a map \dot{s} of the topological space X into the sheaf \mathcal{F} . For an arbitrary U which is open in \mathcal{F} we have

$$\dot{s}^{-1}(U) = \bigcup_{i \in I} \dot{s}_i^{-1}(U).$$

The set $\dot{s}_i^{-1}(U)$ is open in U_i , whence also in X . It follows that $\dot{s}^{-1}(U)$ is an open set in X . Hence $\dot{s} : X \rightarrow \mathcal{F}$ is continuous.

In addition, we have to show that $p \circ \dot{s} = id_X$, where $p : \mathcal{F} \rightarrow X$ is a local homeomorphism. For any $x \in X$ we have an open set U_i containing x . So $p \circ \dot{s}(x) = p \circ \dot{s}_i(x) = p((U_i, s_i)_x) = x$. Thus \dot{s} is a global section of the sheaf \mathcal{F} .

Conversely, a global section of a sheaf defines an atlas.

Let \dot{s} be a global section of a sheaf \mathcal{F} associated to the presheaf F . This means that there exists a continuous map $\dot{s} : X \rightarrow \mathcal{F}$ such that $p \circ \dot{s} = id_X$, where $p : \mathcal{F} \rightarrow X$ is a local homeomorphism. Since \dot{s} is continuous, each point $x \in X$ has an open neighbourhood U_i such that $\dot{s}|_{U_i}$ is continuous. Let $\dot{s}|_{U_i} = \dot{s}_i$. Then we have a continuous map $\dot{s}_i : U_i \rightarrow \mathcal{F}$ defined by $\dot{s}_i(x) = (U_i, s_i)_x$, where $(U_i, s_i)_x$ is the equivalence class of (U_i, s_i) . That is, each $x \in X$ has an open set U_i with $x \in U_i$, and every map \dot{s}_i over U_i defines (U_i, s_i) . Indeed, these (U_i, s_i) 's form the following atlas:

$$\mathcal{U} = \{(U_i, s_i) : i \in I, s_i \in F(U_i)\}.$$

■ An atlas $\mathcal{V} = \{(V_i, t_i) : i \in I\}$ is called a *refinement* of \mathcal{U} if there is a

function $p : J \rightarrow I$, such that, for each index $V_j \subseteq U_{p(j)=i}$ and $s_i|_{V_j} = t_j$, i.e. (V_j, t_j) is sub-chart of (U_i, s_i) , $j \in J$. Moreover, two atlases define the same global section s if and only if they have common refinement [38].

Given a global section s , a pair (U, t) , where $t \in F(U)$, may be called a *chart* for s if either of the two equivalent conditions hold ;

- i) There exist some atlas \mathcal{U} for s with (U, t) as a member.
- ii) For every $x \in U$ the germ of t at x equals $s(x)$.

1.4 Local equivalence relations

Recall that the notion of local equivalence relation on a topological space X was introduced by Grothendieck and Verdier [26] p.485 in series of exercises, presented essentially as open problems for the purpose of constructing a certain kind of topos called an étendue. The concept has been investigated by Rosenthal [51, 52] and more recently by Kock and Moerdijk [38, 39].

Let X be a topological space and let U be an open subset of X . Let $E(U)$ be the set of all

equivalence relations on U . If $V \subseteq U$ is also open, we have a restriction morphism

$$\begin{aligned} E_{UV}: E(U) &\longrightarrow E(V) \\ R &\mapsto R|V = R \cap (V \times V) \end{aligned}$$

defines a functor

$$E: \mathcal{O}(X)^{op} \longrightarrow Sets$$

from the category of open subsets of X and inclusions to the category of sets and functions. Hence E becomes a presheaf on X . This presheaf is denoted by

$$E = \{E(U), E_{UV}, X\}.$$

Rosenthal showed in his paper that this presheaf is in general not a sheaf [51]. In fact, if E is a sheaf, it must satisfy the conditions F_1 and F_2 in first section.

We now give Rosenthal's example.

Example 1.4.1 Let $X \subseteq \mathbb{R}^2$ be defined by

$$X = \{(0, y) : 0 \leq y \leq 1\} \cup \{(x, 1) : 0 \leq x \leq 1\} \cup \{V_n : n \in \mathbb{N}\}$$

where $V_n = \{(1/n, y) : 0 \leq y \leq 1\}$ (See Figure 1).

Let $\{U_i : i \in I\}$ be an open covering of X . We can define an equivalence relation R_i on each open set U_i , for $i \in I$ as follows:

$$xR_iy \Leftrightarrow \text{there is a path in } U_i \text{ joining } x \text{ to } y.$$

Let $E(U_i)$ denotes the set of all equivalence relations on U_i . This lead us to the definition of presheaves defined by equivalence relations as mentioned earlier. Namely,

$$E: \mathcal{O}(X)^{op} \rightarrow Sets$$

is a functor.

Now, let us choose open sets $\{U_1, U_2, U_3\}$ of X such that $U = \bigcup_{i=1}^3 U_i$ as in the following diagram:

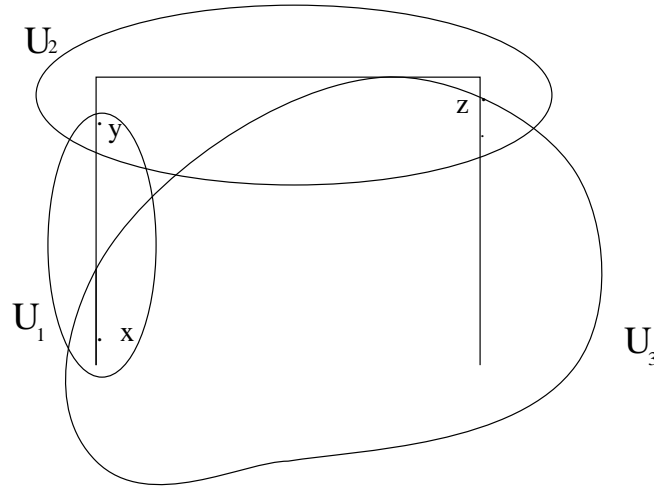


Figure 1

The presheaf E is easily seen to satisfy the condition \mathbf{F}_1 for a sheaf. Now we have to show that presheaf E must satisfy the second condition \mathbf{F}_2 , see previous sections, i.e., there should be an equivalence relation $R \in E(U)$ such that

$$R|U_i = R_i, \text{ for } i = 1, 2, 3.$$

However it does not satisfy F_2 . Suppose we are given such an equivalence relation R on U . Let us choose arbitrary points x, y, z as in the diagram. Since $R|U_1 = R_1$, then there is a path in U joining x to y . Also $R|U_2 = R_2$, then there is a path in U joining y to z . But $x, z \in U_3$, so R_3 should have to satisfy that there is a path in U_3 joining x to z . This is a contraction, because there is no such a path joining them in U_i . So $R|U_3 \neq R_3$. This shows that E is not a sheaf on X .

In first section, we have showed that a sheaf can be constructed for each presheaf. Hence, let $\mathcal{E} \rightarrow X$ denote the sheaf corresponding to E . Let $U \subseteq X$ be open and let $r: U \rightarrow \mathcal{E}$ be a section of \mathcal{E} over U . The set of such sections is denoted by $\Gamma(U, \mathcal{E})$. The set $\Gamma(X, \mathcal{E})$ is the set of global sections of \mathcal{E} . Also Γ is a presheaf.

Definition 1.4.2 A global section r of the sheaf \mathcal{E} is called a *local equivalence relation* on X .

A local equivalence relation r may be given by the following local data which is called an *atlas*: For an open cover $\{U_i : i \in I\}$ of X , if $R_i \in E(U_i)$, $R_j \in E(U_j)$ such that $z \in U_i \cap U_j$, there is an open neighbourhood W of z with $W \subseteq U_i \cap U_j$ and $R_i|W = R_j|_W$. This condition is called the *local compatibility condition*. Conversely, by our earlier discussion, every local equivalence relation r is defined by an atlas which is denoted by $\mathcal{U}_r = \{(U_i, R_i) : i \in I, R_i \in E(U_i)\}$.

In his paper, Rosenthal gives many examples of local equivalence relations. His main example comes from foliation.

Let X be a C^∞ -manifold of dimension n . A *foliation* of codimension q (dimension p , where $p + q = n$) is given by the following;

- (1) an open cover $\{U_i : i \in I\}$ of X .
- (2) submersions $f_i : U_i \rightarrow \mathbb{R}^q$ for $i \in I$. That is, for all $i \in I$, f_i is smooth and for all $x \in U_i$, there is an open neighbourhood V of $f_i(x)$ and a smooth section $g : V \rightarrow U_i$ with $g(f_i(x)) = x$.
- (3) for $i, j \in I$ and $x \in U_i \cap U_j$, there is a diffeomorphism $\gamma_{j,i}$ from a neighbourhood of $f_i(x)$ such that locally $f_i = \gamma_{j,i} f_j$ and if $x \in U_i \cap U_j \cap U_k$, then locally

$$\gamma_{k,i} = \gamma_{k,i} \gamma_{j,i}$$

Example 1.4.3 A foliation on a manifold makes X look locally like n -space divided into parallel p -planes. The simplest example of a foliation is given by the submersion

$$\Pi_2 : \mathbb{R}^p \times \mathbb{R}^{n-p} \longrightarrow \mathbb{R}^{n-p}$$

and every foliation locally looks like this. On each U_i , let R_i be defined by $(x, y) \in R_i$ iff $f_i(x) = f_i(y)$. Condition (3) above is exactly the local compatibility of the equivalence relations. Let U_i and U_j be open sets in X and let R_i and R_j be corresponding equivalence relations on U_i and U_j , respectively. For any $x \in U_i \cap U_j$, there is an open set U_k such that $x \in U_k \subseteq U_i \cap U_j$. Choose two points $x, y \in U_k$ such that $(x, y) \in R_i$, so $f_i(x) = f_i(y)$. By the definition of foliation, $\gamma_{j,i}(f_i(x)) = f_j(x)$ and $\gamma_{j,i}(f_i(y)) = f_j(y)$. Since $\gamma_{j,i}$ is a diffeomorphism, $f_i(x) = f_i(y)$. Then $(x, y) \in R_j$ and so $R_i|_{U_k} = R_j|_{U_k}$. Hence we get a local equivalence relation r on X . For more information about and examples of foliations, see [19, 40, 41, 46, 48, 55].

Suppose we are given topological spaces X and Y , an open cover $\{U_i : i \in I\}$ of X and a family of mapping $\{f_i : i \in I\}$, where $f_i : U_i \rightarrow Y$, and

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every $i, j \in I$. If the mappings are compatible, then the formula $f(x) = f_i(x)$ for $x \in U_i$ defines a function f from the space X into the space Y such that $f|_{U_i} = f_i$ [7].

Example 1.4.4 So the above example, clearly, could be generalised to a space X and a cover $\{U_i\}$ and continuous $f_i : X \rightarrow Y$ for some space Y with the necessary local compatibility. We define an equivalence relation R_i on U_i by $xR_i y$ if and only if $f_i(x) = f_i(y)$. Let R_i and R_j be the equivalence relation on U_i and U_j , respectively, and let $x \in U_k \subseteq U_i \cap U_j$ for $x \in U_i \cap U_j$. By local compatibility condition, $R_i|_{U_k} = R_j|_{U_k}$.

We now give another example of a local equivalence relation using the notion of sober space.

A topological space X is said to be *sober* iff for any open subset $U \in O(X)$ such that

- (i) $U \neq X$
- (ii) for all open W, V in X , if $W \cap V \subseteq U$,

then $W \subseteq U$ or $V \subseteq U$, then there is a unique point $x \in X$ with $U = X - \{x\}$.

Again Rosenthal's paper gives an example of a topological space which is not sober.

This definition is often phrased in terms of closed sets. A closed subset $F \subseteq X$ is called *irreducible* if it can not be written as the union of two smaller closed subsets; that is, whenever F_1 and F_2 are closed sets with $F = F_1 \cup F_2$, then $F_1 = F$ or $F_2 = F$. Clearly, if y is a point of X then $\overline{\{y\}}$ is an irreducible closed set. Thus X is *sober* iff every non empty irreducible closed set is the closure of a unique point.

For observe that, for any open set $U \subseteq X$ and its closed complement $F = X - U$, the set U is proper prime, as in (i) and (ii), iff F is non empty and irreducible.

Any Hausdorff space X is sober; any sober space is T_0 . Sobriety is a weaker separation axiom than T_1 . An important example of sober spaces in algebraic geometry is the Zariski spectrum of a commutative ring [44].

Example 1.4.5 If we take a space X , and an open set $U \subseteq X$, we have the equivalence relation $x \sim y$ iff $\overline{\{x\}}^U = \overline{\{y\}}^U$, where we take closure relative to U . This equivalence relation defines a local equivalence relation on X .

1.5 The Category of Groupoids

The rest of this section is to give some knowledge of groupoid theory, as contained in [11, 29, 43, 46].

An interesting algebraic theory of groupoids exists, and was begun by Brandt and by Baer in the 1920's, well before Ehresmann introduced the concept of groupoid into differential geometry and topology in the 1950's.

We give some references for the algebraic theory of groupoids and their application to problems in group theory see [29], and for general topological groupoids, see [16, 43].

The discussion is carried only as far as is needed to supply a convenient topological framework for the discussion throughout rest of the study.

We begin by reviewing the basic definitions and fixing the notations. Recall that simply a **groupoid** is a small category in which each morphism is an isomorphism. Here we shall give explicit definition of the groupoid and properties.

Definition 1.5.1 A *groupoid* is a category $G = (G, X, \alpha, \beta, m, i)$ given by a set G of arrows, a

set X of objects and four structure maps;

$$G \times_X G \xrightarrow{m} G \begin{array}{c} \xrightarrow{\alpha, \beta} \\ \xleftarrow{i} \end{array} G$$

The maps α and β give for each arrow $g \in G$ its *source* (domain) $\alpha(g)$ and its *target* (codomain) $\beta(g)$. The map m is defined for any pair of arrows f, g with $\alpha(f) = \beta(g)$, and assigns to this pair the composition (f, g) also denoted $f \circ g$. Finally, the map i , called inclusion map, assigns to each object $x \in X$ the identity arrow at x , denoted $i(x)$ (or id_x or 1_x). These maps must satisfy the well-known identities.

$$\alpha(i(x)) = x = \beta(i(x)), \quad (f \circ g) \circ h = f \circ (g \circ h),$$

$$\alpha(f \circ g) = \alpha(g), \quad f \circ i(\alpha(f)) = f,$$

$$\beta(f \circ g) = \beta(f), \quad i(\alpha(f)) \circ f = f,$$

and for each $f \in G$, $f: x \rightarrow y$, there exist an arrow $g: y \rightarrow x$, $g \in G$, so that $f \circ g = i(y)$ and $g \circ f = i(x)$.

Intuitively, one thinks of a groupoid G as a disjoint union of the set of arrows $G(x, y) = \{f \in G \mid f: x \rightarrow y\}$ parameterized by $x, y \in X$. Namely,

$$G = \bigcup_{x, y \in X} G(x, y).$$

Definition 1.5.2 A category $G' = (G', X', \alpha', \beta', m', i')$ is a subgroupoid of a groupoid $G = (G, X, \alpha, \beta, m, i)$ provided G' is a subset of arrows G , and X' is a subset of objects X and its four structure maps are restrictions and G' is a groupoid.

A subgroupoid G' is called *full(wide)* subgroupoid if G' is a full(wide) subcategory. That is, the subgroupoid G' is full(wide) subgroupoid if $G = G'$ ($X = X'$), respectively. A groupoid $G = (G, X, \alpha, \beta, m, i)$ is said to be *transitive* if $G(x, y) = \{f \in G \mid f: x \rightarrow y, \forall x, y \in X\}$ is non-empty and *totally transitive* if the set $G(x, y) = \{f \in G \mid f: x \rightarrow y\}$ has a single element and a groupoid is said to be *locally transitive* if X has a basis of open sets U such that the restriction of G to U is transitive, similarly for *simply transitive* and *locally simple transitive*, so on.

Let $G = (G, X, \alpha, \beta, m, i)$ be a groupoid and let $a \in X$. Let T_a be the full subgroupoid of G on all objects $y \in X$ such that $G(a, y)$ is non-empty. Then, if $x, y \in Ob(T_a)$, $G(x, y)$ is non-empty, since it contains the composite gf for some $f \in G(x, a)$ and some $g \in G(a, y)$. Thus T_a is transitive and is clearly the maximal transitive subgroupoid of G with a , as one of its object T_a is, therefore called the *component* of G containing (or determined by) a [11].

1.5.1 Examples

Example 1.5.3 Any set X may be regarded as a groupoid on itself with $\alpha = \beta = id_X$ and every element an identity. Groupoids in which every element is an identity have been given a variety of names, we will call them **null** groupoids. It has only identities 1_x , one for each $x \in X$, these may be composed with themselves so that $1_x \cdot 1_x = 1_x$ and there are no other composition.

Example 1.5.4 Let X be a set and then there is a groupoid with object set X and set of arrows the product set $X \times X$ so that an arrow $x \rightarrow y$ is simply the ordered pair (y, x) . The composition is then given by

$$(z, y)(y, x) = (z, x).$$

This groupoid looks rather simple, banal and unworthy of consideration. Surprising, though, it plays a key role in the theory and application. One reason is that if G is a subgroupoid of $X \times X$ and G has the same object set X as $X \times X$, then G is essentially an equivalence relation on X . That is, for all $x \in X$; $(x, x) \in G$; if $(x, y) \in G$ then $(y, x) \in G$; and if $(z, y), (y, x) \in G$ then $(z, x) \in G$. Now equivalence relation is a groupoid with the composition above, is important in mathematics and science because they formalize the idea of classification -two elements have the same classification if and only if they are equivalent. In mathematical terms, we say that equivalence relations formalize the idea of quotienting. Thus, it is an important aspect of their applications that groupoids generalise both groups and equivalence relations [10].

Example 1.5.5 Let X be a set and K is a group. We give $X \times X \times K$ the structure of a groupoid on X in the following way, α is the projection onto the second factor of $X \times X \times K$ and β is the projection onto the first factor:

$$\alpha(x, y, g) = y \quad \beta(x, y, g) = x$$

the inclusion map is $x \mapsto 1_x = (x, x, 1)$ and the composition is

$$(z, y, h)(y, x, g) = (z, x, hg)$$

defined iff $y = y'$. The inverse of (y, x, g) is (x, y, g^{-1}) .

Example 1.5.6 [35] A *pointed space* is a pair (X, x) where $x \in X$ and X is a topological space. A *pointed map* $(X, x) \rightarrow (X, y)$ on X is determined by two pointed spaces and a map $\psi: X \rightarrow X$ such that $\psi x = y$. We get a category Top_* of pointed spaces (or spaces with base point). We shall define an equivalence relation in this category: A map $\psi: (X, x) \rightarrow (X, y)$ called a *homotopy*

equivalence if there exists a map $\psi^{-1}: (X, y) \rightarrow (X, x)$ such that $\psi^{-1}\psi \simeq 1_{(X,x)}$ and $\psi\psi^{-1} \simeq 1_{(X,y)}$, where \simeq is the homotopy relation (rel base points). Clearly this relation is an equivalence relation on Top_* . Let $[(X, x); (X, y)]$ denote the set of all homotopy classes of homotopy equivalence maps $(X, x) \rightarrow (X, y)$.

We consider a groupoid over X , called $\mathcal{E}(X)$, such that $\mathcal{E}(X)(x, y) = [(X, x), (X, y)]$ is the set of pointed homotopy classes of homotopy equivalence maps $(X, x) \rightarrow (X, y)$. So the set

$$\mathcal{E}(X) = \bigcup_{x, y \in X} [(X, x); (X, y)]$$

is a groupoid under the composition: $\mathcal{E}(X) \oplus \mathcal{E}(X) \rightarrow \mathcal{E}(X)$, $([\psi], [\psi']) \mapsto [\psi][\psi'] = [\psi\psi']$, where $\mathcal{E}(X) \oplus \mathcal{E}(X) = \{([\psi], [\psi']): \beta[\psi'] = \alpha[\psi]\}$. For any element $[\psi] \in \mathcal{E}(X)(x, y) = [(X, x); (X, y)]$, the source and target maps are $\alpha[\psi] = x$ and $\beta[\psi] = y$ respectively and $\varepsilon: X \rightarrow \mathcal{E}(X)$, $x \mapsto [1_x]$.

Example 1.5.7 An interesting groupoid is the *fundamental groupoid* $\pi_1(X)$ of a topological space X . An object of $\pi(X)$ is a points x of X , and arrow $a: x \rightarrow y$ of $\pi(X)$ is a homotopy class $[a]$ of paths $a: [0, 1] \rightarrow X$ from $x = a(0)$ to $y = a(1)$. Such a path a is a continuous map $I = [0, 1] \rightarrow X$ with $a(0) = x$, $a(1) = x'$ while two paths a, b with the same end points x and x' are homotopic, when there is a continuous map $F: I \times I \rightarrow X$ with $F(t, 0) = b(t)$, $F(t, 1) = a(t)$, and $F(0, s) = x$, $F(1, s) = x'$, for all s and t in I . The composition of paths $a: x \rightarrow x'$ and $b: x' \rightarrow x''$ is the path c which is *a followed by b* given explicitly by

$$h(t) = \begin{cases} a(2t), & 0 \leq t \leq 1/2 \\ b(2t - 1), & 1/2 \leq t \leq 1 \end{cases}$$

Composition applies also to homotopy classes, and makes $\pi(X)$ a groupoid. [11, 46, 42].

Also the category of finite set and bijections, and an individual group can be given as examples of groupoids [12].

Definition 1.5.8 Let $G_1 = (G_1, X_1, \alpha_1, \beta_1, m_1, i_1)$ and $G_2 = (G_2, X_2, \alpha_2, \beta_2, m_2, i_2)$ be two groupoids. A groupoid morphism

$$\phi: G_1 \rightarrow G_2$$

is a functor, i.e., a groupoid morphism between two groupoids G_1 and G_2 is given by two morphisms an objects $Ob(\phi)$ and arrows ϕ (both will be denoted ϕ):

$$Ob(\phi) = \phi: X_1 \rightarrow X_2, \quad \phi: G_1 \rightarrow G_2$$

with respect the structure maps of the groupoid.

$$\phi(i(x)) = i(\phi(x)), \text{ for each } x \in X,$$

$$(\alpha_i(g)) = \alpha_i(\phi(g)), \quad \phi(\beta_i(g)) = \beta_i(\phi(g)), \quad i = 0, 1, \text{ for each } g \in X_1.$$

$$\phi(f \circ g) = \phi(f) \circ \phi(g) \text{ all } f, g \in G_1 \quad \text{with} \quad \alpha_i(f) = \beta_i(g).$$

Definition 1.5.9 Let $G_1 = (G_1, X, \alpha_1, \beta_1, m_1, i_1)$ and $G_2 = (G_2, X, \alpha_2, \beta_2, m_2, i_2)$ be two groupoids over X . A groupoid morphism *over* X or *X -morphism*

$$\phi: G_1 \rightarrow G_2$$

is a functor such that $Ob(\phi) = 1_X$.

Definition 1.5.10 The category of groupoids, denoted by Grd , has as its objects all groupoids $G = (G, X, \alpha, \beta, m, i)$ and its morphisms are functors between them.

For each space X , the category of groupoids over X , denoted by $Grd(X)$, has as its objects groupoids with object set X and X -morphisms as its morphisms.

Definition 1.5.11 A *topological groupoid* is a groupoid $G = (G, X, \alpha, \beta, m, i)$ where G and X are sets equipped with a topology, such that the four structure maps and the inverse map $G \rightarrow G$, $g \mapsto g^{-1}$, assigning to each arrow g its inverse are continuous with respect to these two topologies. A morphism of topological groupoids is a continuous functor between them.

Definition 1.5.12 A topological groupoid G on X is called *étale* if the source $\alpha : G \rightarrow X$ is a local homeomorphism (This implies that all other structure maps are local homeomorphisms, also).

For more information and examples of étale groupoids, we can give Moerdijk' papers such as, see [45, 46, 47]

Definition 1.5.13 A topological groupoid $G = (G, X, \alpha, \beta, m, i)$ is said to be *open* if the source and target maps α, β are both open maps.

We have a category $TGrd$ whose objects are topological groupoids and morphisms are continuous functors.

For each topological space X , the category of topological groupoids over X , denoted by $TGrd(X)$ or $TGrd/X$, has its objects topological groupoid with object X and continuous X -morphism as its morphisms.

Notice that the null topological groupoid X is an initial object and also the topological groupoid $X \times X$ which is called coarse topological groupoid is a final object in the category $TGrd(X)$.

We can give some examples of topological groupoids mentioned earlier. In Example 1.5.3, we consider the set X to be a topological space, then X can be regarded as a topological(étale) groupoid. Also in Example 1.5.4, the groupoid $X \times X$ is a topological groupoid on a topological

space X . In addition, an equivalence relation R is a topological groupoid over the topological space X . The fundamental groupoid $\pi_1(X)$ given in Example 1.5.7 is a topological groupoid in a natural way given certain local conditions on a topological space X [13]. Also each topological group provides an example of a topological groupoid. If $*$: $G \times X \rightarrow X$ is a continuous action of topological group G on a topological space written $*(g, x) = gx$, then the product $G \times X$ has a topological groupoid structure on X [11] with the multiplication $(h, gx)(g, x) = (hg, x)$. Its source and target maps are the second projection and the action itself, respectively.

Chapter 2

Local and Global Subgroupoids

In this chapter, we give the definition of a local subgroupoid of a groupoid G on a topological space X . We show that many ideas generalise smoothly from the local equivalence relations to the local subgroupoids (recently, the concept has been explored widely in [14, 15]). Now here we shall give some of the relations between the local subgroupoids and the local equivalence relations.

Firstly, we give the definition of a local subgroupoid and some main examples.

2.0.2 Local subgroupoids

Let X be a topological space and let G be a groupoid with $Ob(G) = X$. Let U be an open subset of X . Let $L_G(U)$ be the set of all wide subgroupoids of $G|U$, where $G|U$ is the full subgroupoid of G on U , i.e., $G|U = \alpha^{-1}(U) \cap \beta^{-1}(U)$. Let $V, U \subseteq X$ be open sets such that $V \subseteq U$. If H is a wide subgroupoid of $G|U$, then $H|V$ is a wide subgroupoid of $G|V$. So there is a restriction map

$$\begin{aligned} L_{UV}: L_G(U) &\longrightarrow L_G(V) \\ H &\longmapsto H|V. \end{aligned}$$

and these define a presheaf

$$L_G: \mathcal{O}(X)^{op} \longrightarrow Sets.$$

However L_G is not in general a sheaf, as explained in Chapter 1, see also [15]. In the usual way this presheaf L_G defines a sheaf denoted by \mathcal{L}_G . We can define \mathcal{L}_G as follows:

$$\mathcal{L}_G = \bigcup_{x \in X} \mathcal{L}_G^x = \bigcup_{x \in X} \{(U_i, H_i)_x : x \in U_i \subseteq X \text{ open } H_i \in L_G(U_i)\}$$

and

$$p: \mathcal{L}_G \rightarrow X, \quad p(\mathcal{L}_x) = x$$

is the canonical projection, i.e., it is a local homeomorphism. Let $U \subseteq X$ be open and $s: U \rightarrow \mathcal{L}_G$ be a section of \mathcal{L}_G over U . The set of such sections is denoted by $\Gamma(U, \mathcal{L}_G)$ which defines a presheaf:

$$\Gamma: \mathcal{O}(X)^{op} \longrightarrow Sets.$$

The set of global sections of \mathcal{L}_G is denoted by $\Gamma(X, \mathcal{L}_G)$ in Chapter 1. Also every element $H \in L_G(U)$ is associated with a section $s \in \Gamma(U, \mathcal{L}_G)$. If $x \in X$ and $\sigma \in \mathcal{L}_x$, then there are an open neighbourhood $x \in U \subseteq X$ and an $s \in \Gamma(U, \mathcal{L}_G)$ such that

$$\sigma = (U, H)_x = germ_x H = s(x) = s_x,$$

for more detail, see Chapter 1.

Definition 2.0.14 A *local subgroupoid* of a groupoid G on the topological space X is a global section of the sheaf \mathcal{L}_G associated to the presheaf L_G .

In other word, a local subgroupoid of G is a continuous functions from X to \mathcal{L}_G such that $p \circ s = id_X$.

A local subgroupoid s can be defined by atlas as follows. Given an open cover $\{U_i : i \in I\}$ of X , subgroupoids H_i, H_j on U_i, U_j , respectively, such that each point $x \in U_i \cap U_j$ has a neighbourhood W on which H_i and H_j agree. This atlas is denoted by $\mathcal{U}_s = \{(U_i, H_i) : i \in I\}$. We define an equivalence relations on the atlas, namely, H_i and H_j are equivalent at x if there exists W such that $x \in W \subseteq U_i \cap U_j$ and H_i and H_j agree on W . The equivalence classes are called the *germs* of subgroupoid of G at x . These germs form the stalk \mathcal{L}_x^G .

Now we give examples of local subgroupoids.

2.0.3 Examples

Example 2.0.15 Let X be a topological space. Every local equivalence relation on X is a local subgroupoid of the groupoid $X \times X$. For open set U in X , let $E(U)$ be the set all equivalence relations on U . This $E: \mathcal{O}(X)^{op} \rightarrow Sets$ is a presheaf which defines a sheaf \mathcal{E} on X . So we obtain a local equivalence relation r as a global section of \mathcal{E} . It is easy to show that the set $E(U)$ is the family of all wide subgroupoid of $U \times U$. Hence a local equivalence relation r on X is a local subgroupoid of $X \times X$.

Example 2.0.16 Any topological space X can be considered as a groupoid on itself with $\alpha = \beta = id_X$ every element the identity, see Example 1.5.3, called null groupoid. As is well-known, it is an initial object in the category of groupoid on X , $Grd(X)$.

Let us build up the local subgroupoid of X on X . Firstly, we shall construct the corresponding sheaf \mathcal{L}_X . Let $U \subseteq X$ be open, the set

$$L_X(U) = \{U : U \text{ is a wide subgroupoids of } X | U = U\}$$

is just restriction set of U ; $L_X(U) = \{U\}$. For open sets $V, U \subseteq X$ with $V \subseteq U$, the restriction map is

$$L_{UV}^X: L_X(U) \rightarrow L_X(V), \quad U \mapsto V,$$

i.e., $L_{UV}^X(U) = U|V = V$. Hence $L_X: \mathcal{O}(X)^{op} \rightarrow Sets$ is a presheaf. Let \mathcal{L}_X be the sheaf associated with presheaf L_X . Its stalks are such as

$$\mathcal{L}_X^X = \varinjlim_{x \in X} L(U) = \varinjlim_{x \in U} \{U\} = U$$

and the sheaf

$$\mathcal{L}_X = \bigcup_{x \in X} \mathcal{L}_X^X = \bigcup_{x \in X} \{(U, U)_x : x \in U, U \in L_X(U)\}$$

So a local subgroupoid of X is defined by an atlas $\mathcal{U}_s = \{(U_x, U_x) : x \in X\}$.

The following examples give us the relations between local subgroupoids and local equivalence relations.

Example 2.0.17 Let $X^* = \{X_i \mid i \in I\}$ be a partition of a topological space X and let R be an associated equivalence relation of partition on X . Let K be a group, then $G = R \times K$ is a groupoid on X , see Example 1.5.5.

Let $U_i, U_j \subseteq X$ be open sets with $U_j \subseteq U_i$. Let us consider the following structure on X . We define the sets

$$E(U_i) = \{R_i \mid R_i \text{ is an equivalence relation on } U_i\}$$

and similarly

$$L(U_i) = \{H \mid H \text{ is a wide subgroupoid of } G|_{U_i} = R \times K|_{U_i} = R|_{U_i} \times K\}.$$

Clearly, they define functors from \mathcal{O}^{op} to $Sets$, i.e., E and L are presheaves on X . So we define a natural transformation $\gamma: E \rightarrow L$, by $\gamma_U: E(U_i) \rightarrow L(U_i)$, $R_i \mapsto R_i \times K$, for each open set $U_i \subseteq X$. In other word, the following diagram is commutative;

$$\begin{array}{ccc} E(U_i) & \xrightarrow{\gamma_{U_i}} & L(U_i) \\ \mathcal{E}_{U_i U_j} \downarrow & & \downarrow L_{U_i U_j} \\ E(U_j) & \xrightarrow{\gamma_{U_j}} & L(U_j) \end{array} \quad \begin{array}{ccc} R_i & \xrightarrow{\quad} & R_i \times K \\ \downarrow & & \downarrow \\ R_i|_{U_j} & \xrightarrow{\quad} & R_i|_{U_j} \times K = R_i \times K|_{U_j} \end{array}$$

That means that $\gamma: E \rightarrow L$ is a presheaf morphism. It give rise to a sheaf morphism between associated sheaves [5]. Let \mathcal{E}_R and \mathcal{L}_G be the sheaves associated to the presheaves E_R and L_G ,

respectively. Let $\gamma^*: \mathcal{E}_R \rightarrow \mathcal{L}_G$ be the corresponding sheaf morphism, i.e., γ^* is continuous and the following diagram

$$\begin{array}{ccc} \mathcal{E}_R & \xrightarrow{\gamma^*} & \mathcal{L}_G \\ & \searrow p_{\mathcal{E}} & \swarrow p_{\mathcal{L}} \\ & X & \end{array}$$

is commutative.

If r is a local equivalence relation of R given by an atlas $\mathcal{U}_r = \{(U_i, R_i) \mid i \in I\}$, then γ^*r is to be a local subgroupoid of G on X . It can be defined as an atlas $\mathcal{U}_r^* = \{(U_i, R_i \times K) \mid i \in I\}$. In other word, a local equivalence relation of R on a topological space X defines a local subgroupoid of groupoid $R \times K$ on X , while K is a group.

Example 2.0.18 Let G be a groupoid on a topological space X . Then $X \times X$ is also a groupoid on X , see Example 1.5.4.

For $U \subseteq X$ open set, we obtain the following presheaves.

$$L_G = \{L_G(U), L_{UV}, X\} \quad \text{and} \quad E_{X \times X} = \{E(U), E_{UV}, X\}$$

respectively. The set $L_G(U)$ is all wide subgroupoids of $G|U$, $E(U)$ is all equivalence relations on U . These presheaves define the sheaves on X . We have defined a local equivalence relation r on X to be a global section of the sheaf $\mathcal{E}_{X \times X}$ associated with the presheaf $E_{X \times X}$.

Likewise a local subgroupoid s of G is a global section of the sheaf \mathcal{L}_G associated with L_G . Also we can define a morphism of groupoids on X as follows;

$$\Upsilon = [\alpha, \beta]: G \longrightarrow X \times X \quad g \mapsto (\alpha(g), \beta(g))$$

where α and β are the source and target maps, respectively [43]. Let $H \in L(U)$ and $R \in E(U)$ such that $\Upsilon[\alpha, \beta](H) = R$. So we obtain a morphism of groupoids $H \rightarrow R$ on U by $\Upsilon = [\alpha, \beta]$. Hence we define a map $\Upsilon: L(U) \rightarrow E(U)$, $H \mapsto R$, which defines a morphism of presheaves which is a natural transformation of functor. In fact, the diagram

$$\begin{array}{ccc} L_G(U) & \xrightarrow{\Upsilon_U} & E(U) \\ \mathcal{E}_{UV} \downarrow & & \downarrow E_{UV} \\ L_G(V) & \xrightarrow{\Upsilon_V} & E(V) \end{array}$$

is commutative. Then the natural transformation

$$\Upsilon: L_G \longrightarrow E_{X \times X}$$

defines a sheaf morphism

$$\Upsilon^*: \mathcal{L}_G \longrightarrow \mathcal{E}_{X \times X}.$$

i.e., $p_{X \times X} \circ \Upsilon^* = p_G$, where $p_{X \times X}$ and p_G are local homeomorphism on sheaves $\mathcal{E}_{X \times X}$, \mathcal{L}_G , respectively. So we obtain $\Upsilon^*(s) = r$

In this example, the idea leads us to a generalization. Namely, any groupoid morphism gives rise to a sheaf morphism between the corresponding sheaves. In other word, there is a functor from the category of groupoids $Grd(X)$ to the category of sheaves $Sh(X)$.

Suppose given two groupoids G, H and a groupoid morphism $\Phi: G \rightarrow H$ in $Grd(X)$. As usual, we can obtain the following presheaves of sets on X for the groupoids G and H :

$$L_G = \{L_G(U), L_{UV}^G, X\}, \quad L_H = \{L_H(U), L_{UV}^H, X\}$$

for open sets $U, V \subseteq X$ with $V \subseteq U$. That is, L_G and L_H are functors from \mathcal{O}^{op} to $Sets$. Hence the morphism of groupoid $\Phi: G \rightarrow H$ gives rise to a morphism of presheaves and a natural transformation by the following diagram

$$\begin{array}{ccc} L_G(U) & \xrightarrow{\Phi_U^*} & L_H(U) \\ L_{UV}^G \downarrow & & \downarrow L_{UV}^H \\ L_G(V) & \xrightarrow{\Phi_V^*} & L_H(V) \end{array}$$

The natural transformation $\Phi^*: L_G \rightarrow L_H$ is explicitly defined as follows. For each open set $U \subseteq X$, suppose that $A \in L_G(U)$, i.e., if A is a wide subgroupoid of $G|U$, then its image $\Phi_U(A)$ is a wide subgroupoid of $H|U$, since $Ob(\Phi^*)$ is the identity on U . Thus $\Phi_U(A) \in L_H(U)$. Finally we have $\Phi^*(A|U) = \Phi_U(A)|V$.

In sheaf theory, as is well-known, a morphism of presheaves defines a morphism of corresponding sheaves [5, 53, 44].

Let $\mathcal{L}_G, \mathcal{L}_H$ be the sheaves associated with presheaves L_G and L_H , respectively, and also let $\Phi^*: L_G \rightarrow L_H$ be the presheaf morphism as above. Then for each $x \in X$, Φ^* induces a morphism

$$\Phi_x^*: \mathcal{L}_x^G = \lim_{\overrightarrow{x \in U}} L_G(U) \rightarrow \lim_{\overrightarrow{x \in U}} L_H(U) = \mathcal{L}_x^H$$

$$(U, A)_x \rightarrow ((U, \Phi(A)))_x$$

and therefore a map $\Phi^*: \mathcal{L}_G \rightarrow \mathcal{L}_H$ which is a sheaf morphism.

We summarise all knowledge above as follows: Given two groupoids G, H and a groupoid morphism $\phi: G \rightarrow H$ in the category of groupoids on X , $Grd(X)$. Then morphism give rise to a functor \mathcal{L} from $Grd(X)$ to the category of sheaves $Sh(X)$.

Also we obtain a subcategory of $Sh(X)$ whose objects are sheaves defined by groupoids as above and whose morphisms are sheaf morphism. It is denoted by $Sh_{Grd}(X)$ and is a full subcategory of the category $Sh(X)$.

Proposition 2.0.19 *Let $Sh_{Grd}(X)$ be the subcategory of the category $Sh(X)$ as above. Then its initial and final objects are \mathcal{L}_X and $\mathcal{L}_{X \times X}$, respectively.*

Proof: First we shall show that the sheaf \mathcal{L}_X is an initial object of the category $Sh_{Grd}(X)$. Clearly, \mathcal{L}_X is an object of the category $Sh_{Grd}(X)$. Now we have to show that for each object \mathcal{L}_G in $Sh_{Grd}(X)$, there is a unique sheaf morphism between sheaves \mathcal{L}_X and \mathcal{L}_G . That is, there is a unique map $\phi: \mathcal{L}_X \rightarrow \mathcal{L}_G$ such that the following diagram

$$\begin{array}{ccc} \mathcal{L}_X & \xrightarrow{\phi^*} & \mathcal{L}_G \\ & \searrow p_X & \swarrow P_G \\ & X & \end{array}$$

commutes.

Since X is an initial object of the category $Grd(X)$, there is a unique groupoid morphism $I: X \rightarrow G$, for each groupoid G in $Grd(X)$. This morphism give rise to a sheaf morphism $I^*: \mathcal{L}_X \rightarrow \mathcal{L}_G$ with $p_G \circ I^* = p_X$. Therefore I^* must be unique, since I is unique. So $I^* = \phi^*$. Here the sheaf morphism I^* is defined as follows, on each stalks $\mathcal{L}_x^X I_x^*: \mathcal{L}_x^X \rightarrow \mathcal{L}_x^G$, $(U, U)_x \mapsto (U, I(U))_x$.

Likewise the sheaf $\mathcal{L}_{X \times X}$ can be showed to be a final object of the category $Sh_{Grd}(X)$. ■

Category of Local Subgroupoids : LSG(X)

Let $Sh_{Grd}(X)$ be the category of sheaves defined by groupoids as above. We can define a category of local subgroupoids and denote by $LSG(X)$ in which an arrow $\phi^*: s_1 \rightarrow s_2$ is a sheaf morphism $\phi^*: \mathcal{L}_G \rightarrow \mathcal{L}_H$ with $s_1 \circ \phi^* = s_2$ in the category $Sh_{Grd}(X)$, i.e., the following diagram

$$\begin{array}{ccc} \mathcal{L}_G & \xrightarrow{\phi^*} & \mathcal{L}_H \\ & \searrow s_1 & \swarrow s_2 \\ & X & \end{array}$$

commutes, for each groupoid G and H in the category $Grd(X)$. Its class of objects is the set of local subgroupoids of $Grd(X)$.

2.1 Coherent Local Subgroupoids

In this section, we generalise the idea of coherence from local equivalence relations, as given by Rosenthal [52], to local subgroupoids.

We define a partial order structure on the sheaf of germs of the local subgroupoids as follows.

Let s, t be local subgroupoids of G , and let s_x, t_x be their germs at $x \in X$. Then there are open sets U_x and V_x containing x and $H_x \in L_G(U_x)$, $K_x \in L_G(V_x)$ such that H_x defines s_x and K_x defines t_x , i.e.

$$s_x = (U_x, H_x)_x \quad \text{and} \quad t_x = (V_x, K_x)_x.$$

We say $s_x \leq t_x$ if there is an open neighbourhood W of x with $W \subseteq U_x \cap V_x$ and $H_x|W \subseteq K_x|W$, i.e. H_x is a wide subgroupoid of K_x on W . So we obtain a natural order structure on the sheaf of germs of local subgroupoids by the following definition.

Definition 2.1.1 Let s and t be local subgroupoids of G and let $s = (s_x)_{x \in X}$ and $t = (t_x)_{x \in X}$. We write $s \leq t$ iff $s_x \leq t_x$ for all $x \in X$.

Definition 2.1.2 Let $L_G(X)$ be the set of wide subgroupoids of G on X and let $H \in L_G(X)$ and $x \in X$. Then $loc(H)$ is the local subgroupoid defined by

$$loc(H)(x) = (X, H)_x.$$

For the open set U of X , clearly $loc(H)(U) = (U, H|U)$ [15].

Let s be a local subgroupoid of G on X . Globalisation of s is defined by

$$glob(s) = \cap \{H : s \leq loc(H)\}$$

where $H \in L_G(X)$. Thus $s \leq loc(H)$ means that for $x \in X$, $s_x \leq (loc(H))_x$, i.e., if $s_x = (U_x, H_x)_{x \in X}$,

$$H_x|U_x = H_x \subseteq (loc(H))_x|U_x = H|U_x.$$

We think of $glob(s)$ which obtains approximate s by a global subgroupoid. To get a best possible approximate we consider all wide subgroupoids of G on X which locally contain s , and intersect them. That is, $glob(s)$ is the smallest wide subgroupoid of G on X which locally contains s .

Moreover we can provide an alternative useful description of $glob(s)$. Let us suppose that s is given by an atlas $\mathcal{U}_s = \{(U_x, H_x) : x \in X\}$, i.e.,

$$s = (s_x)_{x \in X} = (U_x, H_x)_{x \in X}.$$

Let $\mathcal{V} = \{V_x : x \in X\}$ be an open cover of X such that $x \in V_x \subseteq U_x$ for $x \in X$. Let $H_{\mathcal{V}}$ be the subgroupoid of G generated by $\{H_x|V_x : x \in X\}$. Clearly, $H_{\mathcal{V}} \subseteq G$, because $H_x|U_x = H_x \supseteq H_x|V_x$ and then

$$\bigcup_{x \in X} H_x|V_x = \bigcup_{x \in X} H_x = H_{\mathcal{V}} \supseteq G.$$

Example 2.1.3 Let X be a topological space and let $\mathcal{U} = \{U_x : x \in X\}$ be an open cover of X . We consider the groupoid $G = X \times X$ on X . Then

$$G|U_x = U_x \times U_x = \pi_1^{-1}(U_x) \cap \pi_2^{-1}(U_x)$$

where π_1 and π_2 are the natural projections. But the subgroupoid $H_{\mathcal{U}}$ is generated by $\{G|U_x\}$, i.e.,

$$H_{\mathcal{U}} = \bigcup_{x \in X} G|U_x = \bigcup_{x \in X} (U_x \times U_x)$$

and $H_{\mathcal{U}} \subset G$.

Proposition 2.1.4 Let s be a local subgroupoid of G given by an atlas $\mathcal{U}_s = \{(U_x, H_x) : x \in X\}$. Let $\mathcal{U} = \{U_x : x \in X\}$ and let $H_{\mathcal{V}}$ be the subgroupoid of G generated by $\{H_x|V_x : x \in X\}$. Then

$$glob(s) = \bigcap \{H_{\mathcal{V}} : \mathcal{V} \leq \mathcal{U}\}$$

Proof: Let Q be a subgroupoid of G on X such that $s \leq loc(Q)$. Take an open cover $\{W_x : x \in X\}$ with $x \in W_x \subseteq U_x$ and so $H_x|W_x \subseteq Q|W_x$. Then $H_W \subseteq Q$ and hence $\bigcap \{H_{\mathcal{V}} : \mathcal{V} \leq \mathcal{U}\} \subseteq glob(s)$.

Conversely, if $\mathcal{V} \subseteq \mathcal{U}$, then $H_x|V_x \subseteq H_{\mathcal{V}}|V_x$, since $H_{\mathcal{V}}$ is locally generated by $\{H_x|V_x\}$. Thus $s \leq loc(H_{\mathcal{V}})$. So

$$glob(s) = \bigcap \{H_{\mathcal{V}} : \mathcal{V} \leq \mathcal{U}\}$$

■

We always have $glob(loc(H)) \subseteq H$, for each $H \in L_G(X)$. In fact, we have $loc(H) \subseteq H$, and it follows that $glob(loc(H)) \subseteq glob(H) = H$ [51, 15].

Definition 2.1.5 Let s be a local subgroupoid of G on X .

- i) s is called *coherent* if $s \leq loc(glob(s))$.
- ii) s is called *globally coherent* if $s = loc(glob(s))$
- iii) s is called *totally coherent* if for every open set U , $s|_U$ is coherent.

Definition 2.1.6 Let H be a subgroupoid of G on X , i.e. $H \in L_G(X)$.

- i) H is called *locally coherent* if $\text{loc}(H)$ is coherent.
- ii) H is called *coherent* if $H = \text{glob}(\text{loc}(H))$.

Coherence of s says that in passing between local and global information nothing is lost due to collapsing. Note that for a groupoid H to be coherent we must have that for every open cover $\mathcal{V} = \{V_x : x \in X\}$, $H = H_{\mathcal{V}}$, where $H_{\mathcal{V}}$ is the subgroupoid of H generated by $\{H|_{V_x} : x \in X\}$. We can find many examples for local and global case in Rosenthal's papers [51, 52], if we take a local equivalence relation as a local subgroupoid. (In the local groupoid case have been considered widely in paper 'Local subgroupoids II: examples and properties [15].)

Example 2.1.7 Any topological space X can be considered as a groupoid on itself 1.5.3. Let \mathcal{L}_X be a sheaf corresponding to the groupoid X , see Example 2.0.16. Let s be a local subgroupoid of X . It is easily seen that $\text{loc}(X) = s$, that is, $\text{loc}(X)(U) = (U, X|U) = (U, U)$. Moreover $\text{glob}(s) = X$, since $\text{glob}(s) = \text{glob}(\text{loc}(X)) = X$. So s is globally coherent and X is coherent.

A bundle of groups is a good example of a groupoid. A bundle of groups can also described as a bundle $p: G \rightarrow X$ of spaces in which each fiber $p^{-1}(x)$ is a group in such a way that the resulting operations of addition $+: G \times_p G \rightarrow G$ and inverse $-: G \rightarrow G$ are continuous maps [44].

Example 2.1.8 Clearly a bundle of groups is a groupoid whose source and target maps are equal, i.e. $\alpha = \beta = p$. Let U be an open set in X . The set

$$p^{-1}(U) = G|U = \bigcup_{x \in U} p_x^{-1} = \bigcup_{x \in U} G_x$$

is a groupoid on U . For each open set U in X , we obtain $L_G(U)$ of the set of all subbundles of groups of $G|U$ on U . For $V \subseteq U$ open sets in X ,

$$\begin{aligned} L_{UV}: L_G(U) &\longrightarrow L_G(V) \\ (G|U) &\longmapsto (G|U)|V \end{aligned}$$

is a restriction morphism which defines a presheaf

$$L_G: \mathcal{O}(U)^{op} \rightarrow \text{Sets}.$$

Then we get a sheaf \mathcal{L}_G from the presheaf L . Let s be a local subgroupoid of the bundle of groups G on X . Since $L_G(U) = \{G|U\}$, then s is a globally coherent local subgroupoid of G and G is a coherent groupoid on X . In fact, now, let $\mathcal{V} = \{V_x : x \in X\}$ be an open cover of X such that for each $x \in X$, $x \in V_x \subseteq U_x$, where $\mathcal{U} = \{U_x : x \in X\}$ is also open cover of X . Let $H_{\mathcal{V}}$ be the subgroupoid of G generated by $\{G|V_x : x \in X\}$. $p^{-1}(U_x) = G|U_x \supseteq H_x|V_x$. But $H_{\mathcal{V}} = G$ and $\text{glob}(s) = \cap \{H_V : V \leq U\} = H_{\mathcal{V}} = G$. So $\text{loc}(\text{glob}(s)) = \text{loc}(G) = s$. Hence s is a globally coherent. Since $\text{loc}(G) = s$ and $\text{glob}(\text{loc}(G)) = \text{glob}(s) = G$.

(More examples of local subgroupoids are given in [15].)

We obtain functors loc and glob as follows: Let \mathbf{CL} be the category of coherent local subgroupoid and \mathbf{CG} be the category of locally coherent global subgroupoids on X .

Proposition 2.1.9 *Let $\text{glob}: \mathbf{CL} \longrightarrow \mathbf{CG}$ and $\text{loc}: \mathbf{CG} \longrightarrow \mathbf{CL}$ be functors. These functors form a pair of adjoint functors with loc left adjoint to glob , i.e. $\text{loc} \dashv \text{glob}$.*

Proof: Our categories are $\mathbf{CL} = \{s : s \leq \text{loc}(\text{glob}(s))\}$ and $\mathbf{CG} = \{H : \text{loc}(H) \text{ is coherent}\}$, and natural bijection

$$\theta: \mathbf{CL}(\text{loc}(H), s) \longrightarrow \mathbf{CG}(H, \text{glob}(s))$$

where $H \in \mathbf{CG}$ and $s \in \mathbf{CL}$. If we take $H = \text{glob}(s)$, it gives a unique map

$$\eta: \text{loc}(\text{glob}(s)) \longrightarrow s$$

such that $\theta(\eta) = I$. This map η is a unit of adjunction such that $s \leq \text{loc}(\text{glob}(s))$ and similarly for $\text{loc}(H)$, gives a unique map

$$\xi: H \longrightarrow \text{glob}(\text{loc}(H))$$

The map is dual to the unit of an adjunction is the counit such that $\text{glob}(\text{loc}(H)) \subseteq H$ ■

Under this adjunction, we have an equivalence between globally coherent s and coherent H .

Corollary 2.1.10 *Let $H \in L_G(X)$ and $s = \text{loc}(H)$. Then the transitivity components of H_V are relatively closed and open in the transitivity components of H .*

Proof: Let $M_{x,v}$ and M_x denote the transitivity components of x in H_V and H respectively. Clearly, $M_{x,v} \subseteq M_x$. Because, $s = \text{loc}(H)$, $\text{glob}(s) = \text{glob}(\text{loc}(H)) \subseteq H$, so $H_V \subset H$. Let $y \in M_{x,v}$. Then there is x_1, x_2, \dots, x_n, x , V_1, V_2, \dots, V_{n+1} such that $h_1 \in H|_{V_1}(y, x_1), h_2 \in H|_{V_2}(x_1, x_2), \dots, h_n \in H|_{V_{n+1}}(x_n, x)$, i.e. $g = h_n \dots h_2 \cdot h_1$. Take $V_1 \cap M_x$ and let $z \in V_1 \cap M_x$. Hence $h \in H(z, x)$ and since $k \in H(y, x)$, $h^{-1}k \in H_V(y, z)$. Thus, $h^{-1}k \in H_V$ and $z \in M_{x,v}$. Now let us show that $M_{x,v}$ is closed in M . Let $z \in \overline{M_{x,v}}$ be the closure relative to M_x . For every open neighbourhood U of x , we have V_z , take an element $y \in V_z \cap M_{x,v}$. Then, there is a $g \in H_V(y, x)$ and since $h \in H(z, x)$, we have $h^{-1}g \in H(y, z)$. Since $y, z \in V_z$, $h, g \in H_V(y, z)$ and $z \in M_{x,v}$. Thus $M_{x,v} = \overline{M_{x,v}}$ ■

Theorem 2.1.11 *Let $H \in L_G(X)$ and $s = \text{loc}(H)$. If there is an open neighbourhood W_x of $x \in X$ such that $H|_{W_x}$ has connected transitivity components, then s is coherent.*

Proof: Suppose that s is not coherent. Then, for some $a \in X$, we have $s_a \not\leq \text{loc}(\text{glob}(s))_a$, i.e. given any open neighbourhood W of a , there is a cover $\{V_x : x \in X\}$ and $y, z \in W$ such that there exists an $h \in H(y, z)$, $h \notin H_V(y, z)$. In particular, this is true for W_a . By Corollary 2.1.10, the transitivity component of y in $H|_{W_a}$ is clopen in the transitivity component of y in $H|W_a$, which is connected. This is a contradiction as it forces $h \in H(y, z)$. ■

Corollary 2.1.12 *Let s be a local subgroupoid of G defined by $H_x \in L_G(U_x)$, $x \in X$. Suppose that for every $a \in X$, every open $V \subseteq U_a$ with $a \in V$, there is an open neighbourhood W of a with $W \subseteq V$ and with $H_a|_V$ having connected transitivity component. Then s is totally coherent, i.e. $s|_U$ is coherent for every open U in X .*

Proof: If $a \in U$, where U is open, consider $H_a|_{U \cap U_a}$ and apply the argument from the Theorem 2.1.11. ■

Theorem 2.1.13 *Let $H \in L_G(X)$ and suppose H has connected transitivity components. Then $H = \text{glob}(\text{loc}(H))$. Conversely, if $H = \text{glob}(\text{loc}(H))$ and H has closed transitivity components, then it has connected transitivity components.*

Proof: Given an open cover $\mathcal{V} = \{V_x : x \in X\}$ of X . The subgroupoid H_V generated by $\{H|_{V_x}\}$ is contained in H , $H_V \subseteq H$ and, by Corollary 2.1.10, since transitivity components of H_V are relatively in those of H , which are connected, since M_x is connected, it must be $M_{x,v} = M$ so we have must $H_V = H$, $H = \text{glob}(\text{loc}(H)) = \cap H_V$.

If $H = \text{glob}(\text{loc}(H))$, for every cover \mathcal{V} , $H_V = H$. Let $a \in X$ be such that M_a , the transitivity component of a in H , is not connected. Let U and V be open sets separating M_a . Let $\mathcal{U} = \{U, V, X - \{x\}\}$. Choose $x, y \in M_a$ such that $x \in U \cap M_a$, $y \in V \cap M_a$. Then there exists $g \in H(x, y)$ but $g \notin H_V(x, y)$ since $(U \cap M_a) \cup (V \cap M_a) = M_a$ and they are disjoint, since $H_V \subsetneq H$ we have that $\text{glob}(\text{loc}(H)) \subsetneq H$. This is a contradiction. ■

Proposition 2.1.14 *i) Suppose s is globally and totally coherent on X . If U is open in X , then $s|_U$ is globally coherent.*

ii) If there is an open cover $\{V_x : x \in X\}$ of X such that $s|_{V_x}$ is globally and totally coherent for all $x \in X$, then s is totally coherent.

Proof: i) We have $s = \text{loc}(\text{glob}(s))$. By definition, $\text{glob}(s|_U) \subseteq \text{glob}(s)|_U$, hence $\text{loc}(\text{glob}(s|_U)) \leq \text{loc}(\text{glob}(s)|_U) = \text{loc}(\text{glob}(s))|_U = s|_U$. Since $s|_U$ is coherent, by total coherence of s , we have $s|_U \leq \text{loc}(\text{glob}(s|_U))$. So $s|_U = \text{loc}(\text{glob}(s|_U))$, i.e. $s|_U$ is globally coherence.

ii) By (i), if U is open in X and $s|_{V_x}$ is globally coherence for all $x \in X$, then $s|_{U \cap V_x}$ is globally coherent. Thus $s|_{U \cap V_x} = \text{loc}(\text{glob}(s))|_{U \cap V_x} \leq \text{loc}(\text{glob}(s|_U))|_{V_x}$, since this holds for all $x \in X$, we have $s|_U \subseteq \text{loc}(\text{glob}(s|_U))$, i.e. s is totally coherent. ■

2.1.1 Topological foliations

One of Ehresmann's approaches to the foundations of foliation theory goes via the consideration of a topological space equipped with a further 'fine' topology. Such fine topologies appear also in the context of local equivalence relations and have been considered in [2] and in [51]. We shall need the following elaboration of this idea for the local subgroupoids.

Let s be a local subgroupoid of G on a topological space X which is given by an atlas $\{(U_x, H_x) : x \in X\}$. We can define a new topology on X denoted by X^s . The underlying set of X^s is X . Let $M_{x,a}$ denote the transitivity components of x in U_a for the subgroupoid $H_a \in L_G(U_a)$. Let the topology of X^s be generated by the $M_{x,a}$, $x \in U_a$ and the open sets of X . Then its basic open sets are any set of the form $U \cap M_{x,a}$ where U is open in X , thus this topology is the coarsest for which the original open sets as well as transitive component for H_a are open, and X^s is topologically the disjoint union of the transitive component for H_a , each of them with its subspace topology from X . Since the topology on X^s is finer than that of X , $I: X^s \rightarrow X$, the identity map, is continuous. Hence X^s is a *topological foliation*. The notion of topological foliation was defined by Ehresmann [23].

Theorem 2.1.15 *Let s be a coherent local subgroupoid of the groupoid G on X given by an atlas $\{(U_x, H_x) : x \in X\}$. Then the transitivity components of $\text{glob}(s)$ are connected components of X^s .*

Proof: Let $H = \text{glob}(s)$. Since s is coherent ($s \leq \text{loc}(H)$), for each $a \in X$, choose an open neighbourhood $W_a \subseteq U_a$ such that $H_a|W_a \subseteq H|W_a$. If M is a transitivity component of G , we shall show that

$$M = \bigcup_{a \in M} (M_{a,a} \cap W_a).$$

If $z \in M_{a,a} \cap W_a$ for some $a \in M$, then $h \in H_a(z, a)|W_a$ and hence $h \in G$. Since $a \in M$ and M is the transitivity component of G , then $z \in M$. Hence M is a union of open sets in X^s and so is open.

We prove M is closed in X^s . Let $x \in \overline{M}$, closure is relative to X^s , then $M_{x,x} \cap W_x$ meets M . Let $a \in (M_{x,x} \cap W_x) \cap M$. Then, $k \in H_x|W_x(a, x) \subseteq G|W_x$. Since $a \in M, x \in M$. Thus $M = \overline{M}$ and M is closed in X^s .

Since M is clopen, if it is transitivity connected, we have to show that it is a connected component. Since s is coherent on X and the topology of X^s is finer than that of X , it follows that $s \leq \text{loc}(H)$ in X^s , from which it follows that $\text{glob}(s) \leq \text{glob}(\text{loc}(H))$ and hence $\text{glob}(s) = \text{glob}(\text{loc}(\text{glob}(s)))$, i.e., $H = \text{glob}(s)$ is coherent on X^s . Since its transitivity components are closed by Theorem 2.1.13, they are connected. ■

Let X be a foliated manifold which is defined by submersions $f_i: U_i \rightarrow \mathbb{R}^q$, where $\{U_i : i \in X\}$ is an open cover. Let s denote the associated local equivalence relation. If it is a p -dimensional

foliation, then by declaring $f^{-1}(r)$ open for all i and all $r \in \mathbb{R}^q$, we obtain a p -dimensional manifold structure on X . The $f^{-1}(r)$ are called the leaves of the foliation and the identity map becomes an immersion. Our new manifold is exactly the space X^s and $I: X^s \rightarrow X$ is an immersion. This is how Bourbaki [6] defines a foliated manifold, by specifying X^s and the immersion $I: X^s \rightarrow X$.

If our foliation is given by a single submersion $p: X \rightarrow \mathbb{R}^q$, then $s = \text{loc}(G)$, where $g \in G, g = (x, y)$ iff $p(x) = p(y)$. The groupoid G is the same as the relation of being in the same connected leaf. Thus, by Theorem 2.1.13, $G = \text{glob}(\text{loc}(G))$ and $s = \text{loc}(G)$ is globally coherent. If U is a non empty open set in X , then $f: U \rightarrow \mathbb{R}^q$ is still a submersion and by the above arguments it is easy to see $s \mid U \leq \text{loc}(\text{glob}(s)) \mid U$, and $s = \text{loc}(G)$ is globally and totally coherent. Thus, for our general foliation and hence by Theorem 2.1.15 s is totally coherent.

As a result, some properties of local equivalence relations can be described by the results centred around the notion of local subgroupoids. The interplay of the functors glob and loc says a lot about local subgroupoids on arbitrary topological spaces, (for more examples, see [15]).

Chapter 3

Holonomy groupoid

We quote the following historical remark from [4].

The concept of holonomy groupoid was introduced by C.Ehresmann and Weishu Shih in 1956 [24] and C.Ehresmann in 1961 [23], for a locally simple topological foliation on a topological space X (this means that X has two comparable topologies, and with respect to the finer topology on X , a cover by open sets, in each of which the two topologies coincide). Such a holonomy groupoid is considered as a topological groupoid H on X . It is constructed as a groupoid of local germs of the groupoid H' of holonomy isomorphisms between the transverse spaces U_i of simple open subsets U_i of X such that (U_i, U_{i+1}) is a ‘pure chain’. The holonomy group at $x \in X$ is the vertex group $H(x)$ of H . This holonomy group is isomorphic to the holonomy group $H(y)$ for each y on the same leaf of the foliation as x .

Pradines [49] considered this holonomy groupoid H , in a wider context, with its differential structure. He took the point of view that a foliation determines an equivalence relation R by xRy if and only if x and y are on the same leaf of the foliation, and that this equivalence relation should be regarded as a groupoid in the standard way, with multiplication $(x, y)(y, z) = (x, z)$ for $(x, y), (y, z) \in R$. This groupoid is also written R . In the paracompact case, the locally differential structure which gives the foliation determines a differential structure, not on R itself, but ‘locally’ on R , that is, on a subset W of R containing the diagonal ΔX of X . That is, the foliation determines a locally topological groupoid. The full details of this are given in [18].

This led Pradines to a definition of “un morceau différentiable de groupoïde” G , for which [42], p.161, uses the term “locally differential groupoid”. Pradines’ note [49] asserts essentially that such a (G, W) determines a differential groupoid $Q_0(G, W)$ and a homomorphism $P : Q_0(G, W) \rightarrow G$ such that the “germ” of W extends to a differential structure on G if and only if P is an isomorphism. However his statement of results assumes that the base $X = O_G$ is paracompact and that (G, W) is α -connected. These assumptions seem to be necessary to extend the Globalisation Theorem 2.1 in [4] to the case of germs.

The groupoid $Q_0(G, W)$ is called by Pradines the *holonomy groupoid* of (G, W) .

A construction of the holonomy groupoid in the differential case is attempted by Almeida in [1], using properties of integration of vector fields. However this construction has not been published elsewhere, and of course does not extend to the topological case.

Following Ehresmann's work, there has long been interest in the holonomy group of a leaf of a smooth foliation. For the locally differential groupoid corresponding to a smooth foliation, the vertex groups of the Ehresmann-Pradines holonomy groupoid are the holonomy groups in the standard sense.

The holonomy groupoid H of a smooth foliation on a manifold X was rediscovered (using a different, but equivalent, description) by Winkelkemper [55], as the "graph of the foliation". This was defined as the set S of all triples $(x, y, [\gamma])$, where $x, y \in X$ are on the same leaf L of the foliation, γ is a continuous path on L and $[\gamma]$ is the equivalence class of γ under the equivalence relation \sim which is given by: for the two paths γ_1, γ_2 in L starting at x and ending at y , $\gamma_1 \sim \gamma_2$ if and only if the holonomy of L at x along $\gamma_1^{-1}\gamma_2$ is zero. As pointed out above, these ideas are a special case of the general construction considered here. The way in which the holonomy and monodromy are related in the general case is discussed in [17].

Connes [20] has considered this differential holonomy groupoid H of the foliation and applied to it his general theory of integration based on transverse measures on a measurable groupoid. More recently, in [21], he has applied this and other groupoids in the theory of non commutative C^* -algebras.

Pradines in [49] also defines what he calls a germ of a locally differential groupoid, by saying two locally differential groupoids (G, W) and (G, W') are *equivalent* if there is a third locally differential groupoid (G, W'') such that W'' is an open submanifold of both W and W' . The equivalence classes form the *germ* of (G, W) . Such a germ is called a microdifferential groupoid. His aim is then to define the holonomy groupoid as a functor on the category of such microdifferential groupoids. One of the problems of this theory is that if (G, W) and (G, W') are locally differential groupoids, then $W \cap W'$ may no longer generate G . This difficulty does not occur if the locally differential groupoids are α -connected, since in this case if W generates G and so also does any open subset of W containing O_G . Thus there is still work to be done in investigating examples of these constructions and the relations between and consequences of various possible definitions.

Three principal examples of groupoids are bundles of groups, equivalence relations, symmetry groupoids, and action groupoids associated with an action of a group (or more generally groupoid) on a set (see for example [10]). At present, it seems that only the holonomy of an equivalence relation has been extensively studied, namely in the form of the holonomy groups and holonomy groupoid of a smooth foliation (but see also [51, 52, 38, 48, 14, 15]). There is presumably considerable potential value in the other cases. The paper [15] gives a new range of examples of local subgroupoids which generalise the foliation example – the key idea is that of *star path component of the identities*.

3.1 Local subgroupoids and Locally Topological Groupoids

In this section, we obtain a holonomy groupoid for a certain local subgroupoid by using the idea of locally topological groupoid. Many of the idea of this section derive from in Rosenthal's papers [51, 52]. He obtained a holonomy groupoid of a local equivalence relation on a topological space by using the method of Pradines [49] as sketched by Brown [9].

The construction of the holonomy groupoid is intimately bound up with the properties of the admissible local section of groupoid G . Now, we can give the definition due to Ehresmann [23], but following the notation of [42], with some modifications.

Definition 3.1.1 An *admissible local section* of G is a function $k : U \rightarrow G$ from an open subset U of X such that k satisfies:

- (i) $\alpha k(x) = x$ for all $x \in U$
- (ii) $\beta k(U)$ is open in X , and
- (iii) βk maps U homeomorphically to $\beta k(U)$.

The set U is called the domain of k and denoted by $D(k) = U$.

Let W be a subset of G , and suppose that W has the structure of a topological space with X as a subspace. We say that (α, β, W) has *enough continuous admissible local sections* if for each $w \in W$ there is an admissible local section k of G such that

- (i) $k\alpha(w) = w$,
- (ii) $k(U) \subseteq W$,
- (iii) k is continuous as a function $U \rightarrow W$.

Such a k is called a *continuous admissible local section through w* .

If (α, β, W) has enough continuous admissible section, then (α, β, W) is called *locally sectionable*.

The holonomy groupoid will be constructed for a locally topological groupoid, a term we now define. This definition is a modification of one due to J.Pradines [49] under the name '*un morceau différentiable de groupoïde*'.

Definition 3.1.2 A *locally topological groupoid* is a pair (G, W) consisting of a groupoid G and a topological space W such that

- (G_1) $X = Ob(G) \subseteq W \subseteq G$
- (G_2) $W = W^{-1}$

(G_3) the set $W_\delta = (W \times_\alpha W) \cap \delta^{-1}(W)$ is open in $W \times_\alpha W$ and the restriction to W_δ of the difference map $\delta : G \times_\alpha G \rightarrow G$, $(g, h) \mapsto gh^{-1}$, is continuous.

(G_4) the restriction to W of the source and target maps α and β are continuous, and the triple (α, β, W) is locally sectionable.

(G_5) W generates G as a groupoid.

Note that, in this definition, G is a groupoid but does not need to have a topology. In the cases considered later, G will be a topological groupoid, and W is a subspace, so that condition G_3) is automatic. However, W will usually not be open in G .

Now, we will extend some definitions from local equivalence relations, as given in Rosenthal [52], to the local subgroupoids defined in the previous chapter.

Definition 3.1.3 Let s be a local subgroupoid of a topological groupoid G on X . An atlas $\{(U_x, H_x) : x \in X, H_x \in L_G(U_x)\}$ is called *weakly s -adaptable* if

- (i) the atlas $\{(U_x, H_x) : x \in X, H_x \in L_G(U_x)\}$ locally defines s .
- (ii) $\text{glob}(s)$ is the subgroupoid of G generated by $\{H_x\}_{x \in X}$.

The definition of r -adaptable atlas, (he really in fact used the term ‘family’) which is due to Rosenthal [52] for the case of equivalence relation, includes one more condition. This is the α -connectedness, i.e., each equivalence relation H_x has connected equivalence classes. Then our new the definition leads us to definition of α -connected locally topological groupoid [4]. That is why we do not consider this condition.

Proposition 3.1.4 Let s be a totally coherent local subgroupoid of the topological groupoid G on X . Then s admits a weakly s -adaptable atlas.

Proof: By assumption, we may suppose s is defined by an atlas $\{(U_x, K_x) : x \in X\}$. By coherence, there is an open cover $\mathcal{V} = \{V_x\}_{x \in X}$ of X with $x \in V_x \subseteq U_x$ such that $\text{glob}(s)$ is the a groupoid $K_{\mathcal{V}}$ generated by $\{K_x|_{V_x}\}_{x \in X}$. By corollary 2.1.12, $s|_{V_x}$ is globally coherent hence

$$s|_{V_x} = \text{loc}(\text{glob}(s|_{V_x})).$$

Since $\text{glob}(s) = K_{\mathcal{V}}$ and in some neighbourhood of x , we have $K_x = H_x = \text{glob}(s|_{V_x})$. It follows that Definition 3.1.3,(ii) above will be satisfied for (K_x, V_x) ■

We emphasise that the relationship between this work and Rosenthal’s is that an equivalence relation on X is simply a wide subgroupoid of the groupoid $X \times X$, and we need X to be a topological space to define a local equivalence relation on X . So the appropriate content for this work seems to be that of local subgroupoid of a topological groupoid G on X .

Definition 3.1.5 A local subgroupoid s is said to be *regular* if it is totally coherent and has a weakly s -adaptable atlas $\{(U_x, H_x) : x \in X\}$ such that for all $x \in X$, (α_x, β_x, H_x) is locally sectionable.

Definition 3.1.6 Let s be a local subgroupoid of the topological groupoid G on X . Then we say that s is a *strictly regular* local subgroupoid if it has a regular weakly s -adaptable atlas $\{(U_x, H_x) : x \in X\}$ such that, for each $g \in H_x(x, z)$ and $h \in H_y(x, y)$, then $gh^{-1} \in H_z(y, z)$.

We now give a key construction of a locally topological groupoid from a strictly regular local subgroupoid.

Theorem 3.1.7 *Let G be a topological groupoid on X and s be a strictly regular local subgroupoid of G on X defined by atlas $\mathcal{U}_s = \{(U_x, H_x) : x \in X\}$. Let*

$$H = \text{glob}(s) \quad W = W(\mathcal{U}_s) = \bigcup_{x \in X} H_x.$$

Then (H, W) admit the structure of a locally topological groupoid.

Proof:

(G_1) Because of the definition of H and W , clearly $X \subseteq W \subseteq H$.

(G_2) In fact, $W = W^{-1}$. Let $g \in W$. Then there is an element $x \in X$ such that $g \in H_x$. Since H_x is a groupoid on U_x , $g^{-1} \in H_x$. So $W = W^{-1}$.

(G_3) We will show that $W_\delta = (W \times_\alpha W) \cap \delta^{-1}(W)$ is an open subset in $W \times_\alpha W$. We have to show that, for a base open set $U \times V$ in $G \times_\alpha G$,

$$(U \times V) \cap (W \times_\alpha W) \subseteq \delta^{-1}(W).$$

Let $(k, l) \in (U \times V) \cap (W \times_\alpha W)$. Then $(k, l) \in W \times_\alpha W$. By the definition of W , there exist $x, y \in X$, $k \in H_x(x, z)$, $l \in H_y(x, y)$. Since s is strictly regular, $kl^{-1} \in H_z(y, z)$. This shows that $(k, l) \in \delta^{-1}(W)$. Hence W_δ is an open set in $W \times_\alpha W$.

We now prove the restriction of δ to W_δ is smooth. Since G is a topological groupoid, for each $x \in X$, H_x is a topological groupoid on U_x and so the difference map

$$\delta_x : H_x \times H_x \rightarrow H_x$$

is continuous. Because $H_x \subseteq W$, $x \in X$, using the continuity of the inclusion map $i_x : H_x \rightarrow W$, we get a continuous map

$$i_x \times i_x : H_x \times_\alpha H_x \rightarrow W \times_\alpha W$$

the restriction of W_δ is also continuous, that is,

$$i_x \times i_x : H_x \times_\alpha H_x \rightarrow W_\delta$$

is continuous. Then the following diagram is commutative;

$$\begin{array}{ccc} H_x \times_{\alpha_x} H_x & \longrightarrow & H_x \\ \downarrow i_x \times i_x & & \downarrow i_x \\ W_\delta & \longrightarrow & W \end{array}$$

This verifies (G_3) , since H_x is open in W and hence $H_x \times_{\alpha} H_x$ is open W_δ .

(G_4) We define source and target maps α_W and β_W respectively as follows: if $g \in W$ there exist $x \in X$ such that $g \in H_x$ and we let

$$\alpha_W(g) = \alpha_x(g) \quad \beta_W(g) = \beta_x(g)$$

Clearly α_W and β_W are continuous. Since $\{(U_x, H_x) : x \in X, H_x \in L_G(U_x)\}$ is a regular weakly s -adaptable atlas, so (α_x, β_x, H_x) is locally sectionable, for all $x \in X$. Hence (α_W, β_W, W) is locally sectionable.

(G_5) By definition of weakly s -adaptable atlas, $glob(s)$ is a subgroupoid which is generated by $\{H_x\}_{x \in X}$, then W generates H .

Hence (H, W) is a locally topological groupoid. ■

The following basic example is given in Brown-Mucuk [18]:

Let X be a foliated paracompact manifold and let $\{U_x : x \in X\}$ be a distinguished chart of X . We write R_x for the equivalence relation on U_x given by uR_xv if u, v belong to the same path component of U_x with the leaf topology, i.e., u and v are in the same leaf. This equivalence relation defines a local equivalence relation s on X . So we can get an atlas $\{(U_x, R_x) : x \in X\}$ which locally defines s and $glob(s) = R$. The atlas $\{(U_x, R_x) : x \in X\}$ is weakly s -adaptable. Let

$$W = \bigcup_{x \in X} R_x$$

and let W have its topology as a subspace of $X \times X$. Then $W \subseteq R$ but in general W is not open in R . The triple (α, β, W) has enough continuous admissible sections. So (α_x, β_x, R_x) has enough continuous admissible sections. Hence s is a regular local equivalence relation on X . However the strictly regular condition is proved by using the distinguished chart and paracompactness.

3.2 Holonomy groupoid

There is a main globalisation theorem for a locally topological groupoid. Aof-Brown in [4] stated this main theorem, which shows how a locally topological groupoid gives rise to its holonomy groupoid, which is a topological groupoid satisfying a universal property. This theorem generalises Théoreme 1 of Pradines [49]. Now we state this theorem for certain local subgroupoids.

Theorem 3.2.1 *Let s be a local subgroupoid of a topological space G on X , and suppose given a strictly regular atlas $\{(U_x, H_x) : x \in X, H_x \in L_G(U_x)\}$ for s . Let (H, W) be the associated locally topological groupoid. Then there is a topological groupoid Hol^s , a morphism $\phi : Hol^s \rightarrow H$ of groupoids and an embedding $i : W \rightarrow Hol^s$ of W to an open neighbourhood of $Ob(Hol^s) = X$ such that the following condition are satisfied.*

- (i) ϕ is the identity on objects, $\phi i = id_W$, $\phi^{-1}(W)$ is open in Hol^s , and the restriction $\phi_W : \phi^{-1}(W) \rightarrow W$ is continuous.
 - (ii) if A is a topological groupoid and $\psi : A \rightarrow H$ is a morphism of groupoids such that
 - (a) ψ is the identity on objects,
 - (b) the restriction $\psi_{H_x} : \psi^{-1}(H_x) \rightarrow H_x$ of ψ is continuous and $\phi^{-1}(H_x)$ is open in A , the union of $\phi^{-1}(H_x)$ generates A
 - (c) the triple (α, β, A) is locally sectionable.
- Then there is a unique morphism $\psi' : A \rightarrow Hol^s$ of topological groupoids such that $\phi\psi' = \psi$ and $\psi'a = i\psi a$ for $a \in \psi^{-1}(W)$.

The groupoid Hol^s is called the holonomy groupoid $Hol^s(H, W)$ of the local subgroupoid s .

We now give the construction of holonomy groupoid as in Aof-Brown [4]. Let $H = glob(s)$. Let $\Gamma(H)$ be the set of all local admissible sections of H . Define a product on $\Gamma(H)$ by

$$(tk)(x) = t(\beta k(x))k(x) \quad (*)$$

for two admissible local sections k and t . If k is an admissible local section then write k^{-1} for the admissible local section $\beta k(U) \rightarrow H$, $\beta k(x) \mapsto (k(x))^{-1}$. With this product $\Gamma(H)$ becomes an inverse semigroup. Let $\Gamma^c(W)$ be the subset of $\Gamma(H)$ consisting of admissible local sections which have values in W and are continuous. Let $\Gamma^c(H, W)$ be the subsemigroup of $\Gamma(H)$ generated by $\Gamma^c(W)$. Then $\Gamma^c(H, W)$ is again an inverse semigroup. Intuitively, it contains information on the iteration of local procedures. Let $J(H)$ be the sheaf of germs of all admissible local sections of H . Thus the elements of $J(H)$ are equivalence classes of pairs (x, k) such that $k \in \Gamma(H)$, $x \in U = D(k)$ and (x, k) is equivalent to (y, t) if and only if $x = y$ and k and t agree on a neighbourhood of x . The equivalence class of (x, k) is written $[k]_x$. The product structure on $\Gamma(H)$ induces a groupoid structure on $J(H)$ with X as the set of objects and source and target maps $[k]_x \mapsto x$, $[k]_x \mapsto \beta k(x)$. Let $J^c(H, W)$ be generated as a subgroupoid of $J(H)$ by the sheaf $J^c(W)$ of germs of element of $\Gamma^c(W)$.

Thus an element of $J^c(H, W)$ is of the form

$$[k]_x = [k_n]_{x_n} \dots [k_1]_{x_1}$$

where $k = k_n, \dots, k_1$ with $[k_i] \in J^c(W)$, $x_{i+1} = \beta k_i(x_i)$ $i = 1, \dots, n$ and $x_1 = x \in U = D(k)$.

Let $\psi : J(H) \rightarrow H$ be the final map defined by $\psi([k]_x) = k(x)$, where k is an admissible local section. Then $\psi(J^c(H, W)) = H$ because W generates H . Let $J_o = J^c(W) \cap \text{Ker}\psi$. Then J_o is a normal subgroupoid of $J^c(H, W)$. The holonomy groupoid $Hol^s = Hol(H, W)$ is defined to be the quotient groupoid $J^c(H, W)/J_o$. Let $p : J^c(H, W) \rightarrow Hol^s$ be the quotient morphism and let $p([k]_x)$ be denoted by $\langle k \rangle_x$. Since $J_o \subseteq \text{Ker}\psi$ there is a surjective morphism $\phi : Hol^s \rightarrow H$ such that $\phi p = \psi$.

The topology on the holonomy groupoid Hol^s such that Hol^s with this topology is a topological groupoid is constructed as follows. Let $k \in \Gamma^c(H, W)$ with domain U . A partial function $\sigma_k : W \rightarrow Hol^s$ is defined as follows. The domain of σ_k is the set of $w \in W$ such that $\beta(w) \in U$. A continuous admissible local section f through w is chosen and the value $\sigma_k(w)$ is defined to be

$$\sigma_k(w) = \langle k \rangle_{\beta(w)} \langle f \rangle_{\alpha(w)} = \langle kf \rangle_{\alpha w}$$

Now we prove a Lemma which shows that $\sigma_k(w)$ is independent of the choice of the local section f .

Lemma 3.2.2 *Let $w \in W$, and let s and t be continuous admissible local sections through w . Let $x = \alpha w$. Then $\langle s \rangle_x = \langle t \rangle_x$ in H .*

Proof: By assumption $sx = tx = w$. Let $y = \beta w$. Without loss of generality, we may assume that s and t have the same domain U and have image contained in W . Clearly $st^{-1} \in \Gamma^c(W)$. So $[st^{-1}]_y \in J_o$. Hence in H

$$\langle t \rangle_x = \langle st^{-1} \rangle_y \langle t \rangle_x = \langle s \rangle_x.$$

■ It is proven that $\sigma_k(w)$ is independent of the choice of the local section f

and that these σ_k form a set of charts. Then the initial topology with respect to the charts σ_k is imposed on Hol^s . With this topology H becomes a topological groupoid. The proof is essentially the same as in Aof-Brown [4].

Note that recently the structure given above have been extensively generalised to Lie local subgroupoid and their holonomy and monodromy Lie groupoid [14].

Chapter 4

s-sheaves

A central area in the applications of topology is the relation between local and global properties. Many ideas have been developed for this, including cohomology, sheaves and spectral sequences. More recently, groupoids have played an increasing role.

Our attention will be focused on the interaction of sheaves and groupoids. In this Chapter, we consider the notion of local subgroupoid of a topological groupoid G , which is a global section of a certain sheaf of subgroupoids associated to G . We then construct a category of action of a local subgroupoid.

The background to this idea comes from the notion of local equivalence relation and their relation with certain topos of sheaves called *étendues*, which are categories of sheaves equipped with an action of an étale topological groupoid G . The aim of this chapter is to generalise r -sheaf which obtained from strictly regular open local equivalence relation, to the local subgroupoids case.

To understand all structure from the beginning, now we shall give some basic definitions which are due to Mac Lane and Moerdijk [44].

4.1 Internal Category

Let K be a category with pullbacks. Just as an ordinary small category consists of a set of objects and a set of morphisms, so an *internal category* C in K consist of two objects of K -an ‘object of objects’ C_0 and ‘object of morphisms’ C_1 , together with four arrows of K an arrow m for composition, and three arrows

$$C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \begin{array}{c} \xrightarrow{\alpha, \beta} \\ \xleftarrow{i} \end{array} C_0$$

for domain α , codomain β , and identities i : with the first two, we define the object C_2 of ‘composable pairs’ of morphisms as the pullback

$$\begin{array}{ccc} C_2 = C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ \pi_2 \downarrow & & \downarrow \beta \\ C_1 & \xrightarrow{\alpha} & C_0. \end{array}$$

Indeed, a generalized element $h: X \rightarrow C_1 \times_{C_0} C_1$ is thus just a pair of such elements $f, g: X \rightarrow C_1$ with $\alpha f = \beta g$, that is, ‘a composable pair’. We now require, in addition to the morphisms in above diagram, a fourth morphism in K

$$m: C_2 = C_1 \times_{C_0} C_1 \rightarrow C_1$$

to represent composition of composable pairs. The axioms for an internal category then require, besides the usual identities $\alpha i = \beta i = 1$ and $\alpha m = \alpha \pi_2$, $\beta m = \beta \pi_1$, commutativity of the following two diagrams which express the associative law and the unit law for composition;

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{1 \times m} & C_1 \times_{C_0} C_1 \\ m \times 1 \downarrow & & \downarrow m \\ C_1 \times_{C_0} C_1 & \xrightarrow{m} & C_1 \end{array}$$

$$\begin{array}{ccccc} C_1 \times_{C_0} C_0 & \xrightarrow{1 \times i} & C_1 \times_{C_0} C_1 & \xrightarrow{i \times 1} & C_0 \times_{C_0} C_1 \\ & \searrow \pi_1 & \downarrow m & \swarrow \pi_2 & \\ & & C_1 & & \end{array}$$

These conditions constitute a ‘diagrammatic’ form of the standard definition of a category. If C and D are two such internal categories, then an internal functor $F: C \rightarrow D$ is defined to be a pair of morphisms $F_0: C_0 \rightarrow D_0$ and $F_1: C_1 \rightarrow D_1$ in K making the obvious four squares (with i, α, β, m) commute.

With the evident composition of such functors we have a category $Cat(K)$ with the internal categories in K as objects and internal functors as morphism.

In ordinary category theory, the functors $F: C \rightarrow D$ between two small categories play a role quite different from functors from C into the ambient category - the category of sets. A functor of the latter sort consists as usual of an ‘object function’ $C_0 \rightarrow Sets$ and an arrow ‘functions’

$C_1 \rightarrow \text{Functions}$, suitably related. In this description, we now wish to replace *Sets* by any category K with pullbacks, and C by the internal category (again called C) in K . In order to get a suitable ‘internal’ description of such functors to the universe K , we first reformulate the usual case where the universe is *Sets*. There an object function $F_0 : C_0 \rightarrow \text{Sets}$ can be viewed as a C_0 – indexed family of sets, one for each $x \in C_0$. Just as in the treatment of indexed sets, this C_0 – indexed family can be replaced by a single object over C_0 .

$$p : F \rightarrow C_0$$

where $F = \bigcup_{x \in C_0} F_0(x)$ is the disjoint sum of all the sets $F_0(x)$, and p is the obvious projection. Each set F_0x can then be recovered (up to isomorphism) from p as the fiber $P^{-1}(x)$. Similarly, for the arrow function, each arrow $f : x \rightarrow y$ in C given a map $F_0x \rightarrow F_0y$ of sets, written for $a \in F_0x$ as $a \mapsto f.a$. All these maps, one for each $f \in C_1$, can be described in terms of $p : F \rightarrow C_0$ as one single map specifying the action of any f on any a as

$$\phi : C_1 \times_{C_0} F \rightarrow F, \quad \phi(f, a) = f.a$$

where $C_1 \times_{C_0} F \rightarrow F$ is pullback of p along $\alpha : C_1 \rightarrow C_0$.

By writing down the appropriate diagrams, the preceding description of a functor to *Sets* can be easily generalized to the case of an interval category C in a category K with pullbacks. A C -object in K (also called an ‘internal diagram’ on C) is an object $p : F \rightarrow C_0$ over C_0 equipped with an *action*

$$\phi : C_1 \times_{C_0} F \rightarrow F$$

of C of F , where for this pullback $\alpha : C_1 \rightarrow C_0$ is used to take C_1 an object over C_0 . Here the following diagrams are required to commute:

$$\begin{array}{ccccc} C_1 \times_{C_0} F & \xrightarrow{\phi} & F & C_0 \times F & \xrightarrow{i \times 1} & C_1 \times_{C_0} F \\ \pi_1 \downarrow & & \downarrow p & \searrow \pi_2 & & \downarrow \phi \\ C_1 & \xrightarrow{\beta} & C_0 & & & F \end{array}$$

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 \times_{C_0} F & \xrightarrow{1 \times \phi} & C_1 \times_{C_0} F \\ m \times 1 \downarrow & & \downarrow \phi \\ C_1 \times_{C_0} F & \xrightarrow{\phi} & F \end{array}$$

(The second and third express the unit and associativity laws for the action.)

If $F = (F, p, \phi_1)$ and $G = (G, q, \phi_2)$ are two such C -object in K , a *morphism* of C -objects from F to G is simply a morphism $\psi : F \rightarrow G$ in K which preserves the structure involved. In term of diagram, it means that

$$\begin{array}{ccc}
 F & \xrightarrow{\eta} & G \\
 & \searrow p & \swarrow q \\
 & C_0 &
 \end{array}
 \quad
 \begin{array}{ccccc}
 C_1 \times_{C_0} C_1 \times_{C_0} F & \xrightarrow{1 \times \phi} & C_1 \times_{C_0} F \\
 m \times 1 \downarrow & & \downarrow \phi \\
 C_1 \times_{C_0} F & \xrightarrow{\phi} & F
 \end{array}$$

are required to commute.

4.2 G-sheaves

Let

$$\mathbf{C} = \begin{array}{ccccc}
 & \xrightarrow{\pi_1} & & & \\
 C_2 & \xrightarrow[\pi_2]{m} & C_1 & \xrightarrow[\beta]{\alpha} & C_0 \\
 & \xrightarrow{\quad} & & &
 \end{array} \quad (*)$$

be a topological internal category, where C_0 is the space of objects, C_1 is the space of morphism and C_2 is the space of composable pairs of morphisms, m represents composition, α and β are the domain and codomain maps, respectively, which have a common section $i : C_0 \rightarrow C_1$.

Definition 4.2.1 Let \mathbf{C} be a topological category. A \mathbf{C} -Sheaf is a sheaf $p : \mathcal{F} \rightarrow X$ together with a diagram

$$\begin{array}{ccc}
 C_1 \times \mathcal{F} & \xrightarrow{\phi} & \mathcal{F} \\
 \pi_1 \downarrow & & \downarrow p \\
 C_1 & \xrightarrow{\beta} & C_0
 \end{array} \quad (**)$$

which is commutative and furthermore:

$$\begin{array}{ccc}
 C_0 \times \mathcal{F} & \xrightarrow{i \times I} & C_1 \times \mathcal{F} \\
 \pi_2 \downarrow & \searrow \phi & \\
 \mathcal{F} & &
 \end{array}
 \quad
 \begin{array}{ccccc}
 C_1 \times C_1 \times \mathcal{F} & \xrightarrow{1 \times \phi} & C_1 \times \mathcal{F} \\
 m \times I \downarrow & & \downarrow \phi \\
 C_1 \times \mathcal{F} & \xrightarrow{\quad} & \mathcal{F}
 \end{array}$$

The first and second express respectively the unit and associativity laws for the map ϕ , i.e., ϕ is an action of \mathcal{F} on X [44].

A \mathbf{C} -Sheaf morphism is a sheaf map which preserves the actions. In this way, we obtain the category $Sh(X : \mathbf{C})$ of \mathbf{C} -sheaves.

Now, let G be a topological groupoid on X . If we take $G = C_1$ and $G \times G = C_2$ in (\star) we obtain a \mathbf{G} -topological category. In this category, α and β are source and target map, m represents composition, π_1 and π_2 are the canonical projections and ϵ is the object map.

$$\mathbf{C} = G \times_X G \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \\ \xrightarrow{m} \end{array} G \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} X$$

A sheaf $p: \mathcal{F} \rightarrow X$ is called \mathbf{G} -sheaf if it satisfies (**), unit and associativity laws above, i.e., let us given following diagram as in (**),

$$\begin{array}{ccc} G \times_X \mathcal{F} & \xrightarrow{\phi} & \mathcal{F} \\ \pi_1 \downarrow & & \downarrow p \\ G & \xrightarrow{\beta} & X \end{array}$$

where $G \times_X \mathcal{F}$ is a pullback on β . Then, it satisfies

- (i) $p(\phi(g)(e_x)) = y$, for $g \in G(x, y)$
- (ii) $\phi((g_x)(e_x)) = e_x$, for $e_x \in \mathcal{F}_x$
- (ii) $\phi(h)(\phi(g)(e_x)) = \phi(k)(e_x)$, where $g_x \in G(x, x), g \in G(x, y), h \in G(y, z)$ and, $k = gh \in G(x, z)$.

So each element g in $G(x, y)$ defines a morphism

$$g_{\#}: \mathcal{F}_x \rightarrow \mathcal{F}_y$$

of stalks such that $I_{\#} = I$ and $(hg)_{\#} = g_{\#}h_{\#}$. Thus an action of G on \mathcal{F} defines a functor

$$\mathcal{F}: G \rightarrow \{\text{stalks of sheaves.}\}$$

The map ϕ is called a *transport* along G in \mathcal{F} .

We give as an example of G -sheaf by taking an equivalence relation R on X . Then

$$\mathbf{R} = R \times R \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \\ \xrightarrow{m} \end{array} R \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \end{array} X \quad (\star)$$

is an internal topological category. Since R is a topological groupoid, it can be shown that an R -sheaf can be described as a sheaf $p: \mathcal{F} \rightarrow X$ with an equivalence relation S on \mathcal{F} such that if $(x_1, x_2) \in R$ and $e_1 \in p^{-1}(x_1)$, there is a unique $e_2 \in p^{-1}(x_2)$ with $(e_1, e_2) \in S$.

Proposition 4.2.2 *The category of \mathbf{G} -sheaves is a Grothendieck topos.*

Proof: See Moerdijk [45]. ■ This topos is denoted by BG , and called the *classifying topos* of G . The definition of the topos BG also makes sense if G is just a continuous category (a category object in *Locales*) rather than a groupoid. We can find a lot of examples of the topos BG in [45].

4.3 r -sheaf

Let $p : \mathcal{F} \rightarrow X$ be a sheaf over X , and let U be open in X . Let $Q(U, \mathcal{F})$ consist of pairs (R_U, S_U) , where R_U and S_U are equivalence relation on U , $\mathcal{F}|_U$, respectively, such that $p : \mathcal{F}|_U \rightarrow U$ is compatible with R_U and S_U i.e.

$$(e_1, e_2) \in S_U \text{ implies } (p(e_1), p(e_2)) \in R_U$$

and we have the following pullback

$$\begin{array}{ccc} \mathcal{F}|_U & \longrightarrow & (\mathcal{F}|_U)/S_U \\ p \downarrow & & \downarrow q \\ U & \longrightarrow & U/R_U \end{array}$$

with q a local homeomorphism. This implies that if $p(e) = x$ and $x' \in [x]$, the equivalence class of x with respect to R_U , there is a unique

$$e' \in p^{-1}(x) \quad \text{with} \quad (e', e) \in S_U.$$

We can give the following theorem.

Theorem 4.3.1 *Let \mathcal{F} be a sheaf on X . Let $\mathcal{O}(X)$ denote the open sets of X . Then*

$$Q(-, \mathcal{F}) : \mathcal{O}(X)^{op} \rightarrow \mathbf{Sets}$$

is a presheaf.

Proof: See Rosenthal [52]. ■

Let $Q_{\mathcal{F}}$ denote the associated sheaf. We have a forgetful functor of sheaves $Q_{\mathcal{F}} \rightarrow \mathcal{E}$, the sheaf of local equivalence relation on X .

Let r be a local equivalence relation on X .

Definition 4.3.2 An r -structure on a sheaf \mathcal{F} is a local equivalence relation t on \mathcal{F} such that (r, t) is a global section of $Q_{\mathcal{F}}$, i.e., $p(t) = r$.

Definition 4.3.3 An r -sheaf on X is a pair (\mathcal{F}, t) , where \mathcal{F} is a sheaf on X and t is an r -structure on \mathcal{F} .

If (\mathcal{F}_1, t_1) and (\mathcal{F}_2, t_2) are r -sheaves, an r -sheaf morphism is a sheaf map $\mathcal{F}_1 \rightarrow \mathcal{F}_2$, which locally preserves the r -structures. Thus we have a category $Sh(X; r)$ of r -sheaves.

It is clear that because of $Sh(X : R) \cong Sh(X : R/X)$ [52], every R -sheaf can be viewed as an r -sheaf. It is shown that every r -sheaf can be made into R -sheaf. If $p : \mathcal{F} \rightarrow X$ is an r -sheaf with an r -structure t , we must produce an R -action $\phi : R \times \mathcal{F} \rightarrow \mathcal{F}$ making the following diagram commute:

$$\begin{array}{ccc} R \times \mathcal{F} & \xrightarrow{\phi} & \mathcal{F} \\ \pi_1 \downarrow & & \downarrow p \\ R & \xrightarrow{\beta} & X. \end{array}$$

4.4 s -sheaf

In this section, we shall define s -sheaf for a local subgroupoid $s : X \rightarrow \mathcal{L}_G$ given by an atlas $\mathcal{U}_s = \{(U_i, H_i) : i \in I, H_i \in L_G(U_i)\}$. Let \mathcal{F} be a sheaf on X . For any open subset $U \subseteq X$, we consider the set $I_{\mathcal{F}}(U)$ consisting of pairs (H_i, ϕ_i) , where $H_i \in L_G(U_i)$ and ϕ_i is a transport by H_i on $\mathcal{F}|_{U_i} = p^{-1}(U_i)$. For $U_j \subseteq U_i$, there is a restriction map

$$I_{\mathcal{F}}(U_i) \longrightarrow I_{\mathcal{F}}(U_j),$$

and this give the presheaf $I_{\mathcal{F}}$. Furthermore, there is a forgetful functor $I_{\mathcal{F}} \rightarrow L_G$ given by $(H_i, \phi_i) \mapsto H_i$. Let $\mathcal{I}_{\mathcal{F}}$ be the associated sheaf, so the forgetful functor $I_{\mathcal{F}} \rightarrow L_G$ induces a map of sheaves $\mathcal{I}_{\mathcal{F}} \rightarrow \mathcal{L}_G$. Fix a local subgroupoid s of G on X .

Definition 4.4.1 An s -transport on the sheaf \mathcal{F} is a global section t of $\mathcal{I}_{\mathcal{F}}$ such that $p(t) = s$. An s -sheaf on X is a sheaf on X together with an s -transport.

The notation of the transport preserving map between two s -sheaves on X can be defined as follows;

Definition 4.4.2 Let G be a topological groupoid on X and let $\mathcal{F}_1, \mathcal{F}_2$ be s -sheaves with transports ϕ_1 and ϕ_2 , respectively. An s -sheaf morphism from \mathcal{F}_1 to \mathcal{F}_2 is a sheaf map $\eta : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ such that the following diagram is commutative:

$$\begin{array}{ccc}
G \times \mathcal{F}_1 & \xrightarrow{I \times \eta} & G \times \mathcal{F}_2 \\
\downarrow \phi_1 & & \downarrow \phi_2 \\
\mathcal{F}_1 & \xrightarrow{\eta} & \mathcal{F}_2
\end{array}$$

Let $Sh(X; s)$ denote the category of s -sheaves and s -sheaf morphisms. There exists a faithful functor from $Sh(X; s)$ to $Sh(X)$. Because every s -sheaf on X is a sheaf on X .

From the definitions, it also follows that the property of being an s -sheaves is locally property, i.e., if the base space X is covered by open sets U such that the restriction of the sheaf $\mathcal{F} \rightarrow X$ to each U is an $s|U$ -sheaf, then \mathcal{F} is an s -sheaf.

Corollary 4.4.3 *Let s be a locally transitive local subgroupoid of G on X . Then any sheaf \mathcal{F} has at most one s -transport.*

Proof: The proof is similar to Theorem 2.2 given in [38]. ■

Note that it has been shown that a local equivalence relation gives rise to an étendue, which is a particular kind of topos, and that Kock and Moerdijk [39] have shown that every étendue arises from a local equivalence relation. This suggests the problem of characterising the topos of s -sheaves arising from a local subgroupoid s .

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